L^p Theory for the div-curl System

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Abstract
The purpose of this paper is to solve the div-curl system in the framework of L^p theory. We shall give some sufficient conditions for the existence of solutions, and give the estimates of the solutions by the given data. We can also obtain the uniqueness of solution in the particular case of the domain.

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1 Introduction

In this paper, we shall consider the following div-curl system.
\[
\text{curl } u = B, \quad \text{div } u = f \quad \text{in } \Omega, \quad \nu \cdot u = g \quad \text{on } \partial \Omega, \quad (1.1)
\]
or
\[
\text{curl } u = B, \quad \text{div } u = f \quad \text{in } \Omega, \quad \nu \times u = h \quad \text{on } \partial \Omega \quad (1.2)
\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^3\) which may be a multi-connected and have holes satisfying some regularity conditions and \(\nu\) is the unit outer normal vector field on the boundary \(\partial \Omega\), and a vector field and a function \(B, f\) are given in \(\Omega\), and a vector field and a function \(h, g\) are given on \(\partial \Omega\), respectively.

We shall consider the existence of solution for the problem (1.1) or (1.2) and estimate of solution by the given data. In the framework of \(L^2\) theory
and $C^\alpha$ Schauder theory, Pan [10] gave the existence and the estimate in $L^2$ theory and $C^\alpha$ Schauder theory. About the estimates, there are many articles. For example, see Aramaki [3], Bates and Pan [4], Dautray and Lions [7] for $L^2$ estimate, and Bolik and Wahl [5] and Wahl [11] for $C^\alpha$ Schauder estimate.

On the contrary, in the framework of $L^p$ theory it seems that there are few literatures (cf. Amrouche and Seloula [1, 2]). Thus we shall give the existence and estimate of solutions for the problem (1.1) or (1.2) in the framework of $L^p$ theory. Main theorems will be given in section 3 (Theorem 3.3 and Theorem 3.5).

2 Preliminaries

In this section, we shall give the preliminaries for the div-curl system in the next section. First, we assume the regularity and shape of the domain $\Omega$. Next, we give the well definedness of the trace to the boundary of $\Omega$ for some fields, and then some estimates.

Throughout this paper we assume the following (O1) and (O2) with respect to the regularity and shape of the domain $\Omega$.

(O1) $\Omega$ is a bounded domain in $\mathbb{R}^3$ with $C^{r,1}$ ($r \geq 1$) boundary $\Gamma = \partial \Omega$ of dimension 2 and $\Omega$ is locally situated on one side of $\Gamma$. $\Gamma$ has a finite number of connected components $\Gamma_1, \ldots, \Gamma_{m+1}$ where $m \geq 0$ and $\Gamma_{m+1}$ denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(O2) There exist $n$ manifolds of dimension 2 and of class $C^4$ denoted by $\Sigma_1, \ldots, \Sigma_n$ ($n \geq 0$) such that $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$ and non-tangential to $\Gamma$, and $\hat{\Omega} := \Omega \setminus \Sigma$, where $\Sigma = \bigcup_{i=1}^n \Sigma_i$, is simply connected and pseudo-$C^{1,1}$.

Here according to [1], it is said that the bounded open set $D$ in $\mathbb{R}^3$ is called pseudo-$C^{1,1}$ if for any point $x$ on the boundary $\partial D$ there exists an integer $r(x)$ equal to 1 or 2 and a neighborhood of $V_x$ of $x$ such that $D \cap V_x$ has $r(x)$ connected components, each boundary being of $C^{1,1}$.

For $1 \leq p < \infty$, if we define

$$\mathbb{H}^p_1(\Omega) = \{ u \in L^p(\Omega, \mathbb{R}^3); \text{div} u = 0, \text{curl} u = 0 \text{ in } \Omega, \nu \cdot u = 0 \text{ on } \Gamma \},$$

then $\dim \mathbb{H}^p_1(\Omega, \mathbb{R}^3) = n$ which is called the first Betti number. We note that $\hat{\Omega}$ is of class $C^{1,1}$. We consider the following problem: for every $i = 1, \ldots, n$,

$$\begin{cases}
-\Delta p = 0 & \text{in } \hat{\Omega}, \\
\frac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma, \\
[p]_{\Sigma_k} = \text{constant}, \left[ \frac{\partial p}{\partial \nu} \right]_{\Sigma_k} = 0 & \text{for } k = 1, \ldots, n \\
\int_{\Sigma_k} \frac{\partial p}{\partial \nu} dS = \delta_{ik} & \text{(k = 1, \ldots, n)}
\end{cases}$$
where $[\cdot]_{\Sigma_k}$ is the jump of a function across $\Sigma_k$ and $\langle \cdot, \cdot \rangle_{\Sigma_k}$ denotes the duality of $W^{1-1/q,q}(\Sigma_k)$ and its dual space for any $1 < q < \infty$. Since $\hat{\Omega}$ is of $C^{1,1}$ class, it follows from [1] and Girault and Raviart [8] that this problem has a unique solution $p_i \in W^{2,q}(\hat{\Omega})$ for any $1 < q < \infty$. Since $\nabla p_i \in W^{1,q}(\Omega, \mathbb{R}^3) \subset L^q(\Omega, \mathbb{R}^3)$, we can extend to a field on $\Omega$ as a field of $L^q(\Omega, \mathbb{R}^3)$, which is denoted by $\nabla p_i$. Then $\{\nabla p_i\}_{j=1,\ldots,n}$ is a basis of $\mathbb{H}^1_q(\Omega)$. See [1, Corollary 4.1].

Next we define

$$\mathbb{H}_2^p(\Omega) = \{ u \in L^p(\Omega, \mathbb{R}^3); \text{div} u = 0, \text{curl} u = 0 \text{ in } \Omega, \mathbf{\nu} \times u = 0 \text{ on } \Gamma \},$$

then $\dim \mathbb{H}_2^p(\Omega) = m$ which is called the second Betti number. For every $j = 1, \ldots, m$, the following problem:

$$\begin{cases}
-\Delta q = 0 & \text{on } \Omega, \\
q|_{\Gamma_{m+1}} = 0, q|_{\Gamma_j} = \text{constant for } j = 1, \ldots, m, \\
\int_{\Gamma_j} \frac{\partial q}{\mathbf{\nu}} dS = \delta_{jk} \text{ for } k = 1, \ldots, m, \int_{\Gamma_{m+1}} \frac{\partial q}{\mathbf{\nu}} dS = -1
\end{cases}$$

has a unique solution $q_j \in W^{1,q}(\Omega)$ for any $1 < q < \infty$. Then $\{\nabla q_j\}_{j=1,\ldots,m}$ is a basis of $\mathbb{H}_2^p(\Omega)$. See [1, Corollary 4.2].

If $n = 0$, we say that $\Omega$ is simply connected, and if $m = 0$ we say that $\Omega$ has no holes.

Let $1 < p < \infty$ and define a space

$$\mathcal{H}^p(\Omega, \text{curl}) = \{ u \in L^p(\Omega, \mathbb{R}^3); \text{curl} u \in L^p(\Omega, \mathbb{R}^3) \},$$

then the space is a Banach space with respect to the norm

$$\| u \|_{\mathcal{H}^p(\Omega, \text{curl})} = \| \text{curl} u \|_{L^p(\Omega)} + \| u \|_{L^p(\Omega)},$$

and define a space

$$\mathcal{H}^p(\Omega, \text{div}) = \{ u \in L^p(\Omega, \mathbb{R}^3); \text{div} u \in L^p(\Omega) \},$$

then the space is a Banach space with respect to the norm

$$\| u \|_{\mathcal{H}^p(\Omega, \text{div})} = \| \text{div} u \|_{L^p(\Omega)} + \| u \|_{L^p(\Omega)},$$

moreover if we define

$$\mathcal{H}^p(\Omega, \text{curl}, \text{div}) = \mathcal{H}^p(\Omega, \text{curl}) \cap \mathcal{H}^p(\Omega, \text{div}),$$

the space is also a Banach space with respect to the norm

$$\| u \|_{\mathcal{H}^p(\Omega, \text{curl}, \text{div})} = \| \text{curl} u \|_{L^p(\Omega)} + \| \text{div} u \|_{L^p(\Omega)} + \| u \|_{L^p(\Omega)}.$$
Lemma 2.1. Let $1 < p < \infty$. Then the following holds.

(i) The normal trace map $C^\infty(\overline{\Omega}, \mathbb{R}^3) \ni u \to \nu \cdot u \mid \Gamma$ can be extended to a continuous linear operator from $H^p(\Omega, \text{div})$ to $W^{-1/p,p}(\Gamma)$.

(ii) The tangential trace map $C^\infty(\overline{\Omega}, \mathbb{R}^3) \ni u \to u_T = u - (\nu \cdot u)\nu = (\nu \times u) \times \nu$ can be extended to a continuous linear operator from $H^p(\Omega, \text{curl})$ to $W^{-1/p,p}(\Gamma, \mathbb{R}^3)$.

For the proof, see Chen and Pan [6].

Thus for $u \in H^p(\Omega, \text{div})$, $(\nu \cdot u, 1)_{\Gamma_j}$ ($j = 1, \ldots, m + 1$) are well defined where $(\cdot, \cdot)$ denotes the duality of $W^{-1/p,p}(\Gamma_j)$ and $W^{1/p',p'}(\Gamma_j)$, $p'$ is the conjugate number of $p$ that is determined by $1/p + 1/p' = 1$. Similarly, for $u \in H^p(\Omega, \text{curl})$, $(\nu \times u, 1)_{\Gamma_j}$ ($j = 1, \ldots, m + 1$) are well defined.

Here we give the curl-free or divergence-free lifting.

Lemma 2.2. Let $k \geq 0$ be an integer and let $1 < p < \infty$. If $\Omega \subset \mathbb{R}^3$ is a bounded domain with $C^{k+1,1}$ boundary $\Gamma$ and $g \in W^{k+1-1/p,p}(\Gamma)$, there exists $u \in W^{k+1,p}(\Omega, \mathbb{R}^3)$ such that curl $u = 0$ in $\Omega$ and $\nu \cdot u = g$ on $\Gamma$, and there exists a constant $C = C(p,\Omega)$ such that

$$
||u||_{W^{k+1,p}(\Omega)} \leq C ||g||_{W^{k+1-1/p,p}(\Gamma)}.
$$

Proof. Define $f = \frac{1}{|\Omega|} \int_{\Gamma} g dS = \text{constant}$ where $dS$ is the surface area of $\Gamma$, then we have

$$
||f||_{W^{k,p}(\Omega)} = ||f||_{L^p(\Omega)} \leq C(\Omega) ||g||_{L^p(\Gamma)}.
$$

We consider the equation

$$
\begin{cases}
\Delta \phi = f & \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} = g & \text{on } \Gamma.
\end{cases}
$$

Since $\int_{\partial \Omega} f dx = \int_{\Gamma} g dS$, the equation has a unique solution $\phi \in W^{k+2,p}(\Omega)$ up to an additive constant, and

$$
||\phi||_{W^{k+2,p}(\Omega)} \leq C(||f||_{W^{k,p}(\Omega)} + ||g||_{W^{k+1-1/p,p}(\Gamma)}) \leq C_1 ||g||_{W^{k+1-1/p,p}(\Gamma)}.
$$

Define $u = \nabla \phi \in W^{k+1,p}(\Omega)$. Then curl $u = 0$ in $\Omega$ and $\nu \cdot u = \nu \cdot \nabla \phi = g$ on $\partial \Omega$, and

$$
||u||_{W^{k+1,p}(\Omega)} = ||\nabla \phi||_{W^{k+1,p}(\Omega)} \leq C ||\phi||_{W^{k+2,p}(\Omega)} \leq C_1 ||g||_{W^{k+1-1/p,p}(\Gamma)}.
$$

Lemma 2.3. Let $k \geq 0$ be an integer and let $1 < p < \infty$. If $\Omega \subset \mathbb{R}^3$ is a bounded domain with $C^{k+1,1}$ boundary $\Gamma$ and $h \in W^{k+1-1/p,p}(\Gamma)$ satisfying $\nu \cdot h = 0$ on $\Gamma$, then there exists $u \in W^{k+1,p}(\Omega, \mathbb{R}^3)$ such that $\text{div } u = 0$ in $\Omega$ and $u_T = h$ on $\Gamma$, and there exists a constant $C = C(p,\Omega)$ such that

$$
||u||_{W^{k+1,p}(\Omega)} \leq C ||h||_{W^{k+1-1/p,p}(\Gamma)}.
$$

Proof. We can choose $v \in W^{k+1,p}(\Omega, \mathbb{R}^3)$ such that $v = h$ on $\Gamma$ and satisfies $\|v\|_{W^{k+1,p}(\Omega)} \leq C(\Omega)\|h\|_{W^{k+1-1/p,p}(\Gamma)}$. We consider the equation

$$
\begin{cases}
\Delta \varphi = \text{div} \, v & \text{in } \Omega, \\
\varphi = 0 & \text{on } \Gamma.
\end{cases}
$$

Since $\text{div} \, v \in W^{k,p}(\Omega)$, this equation has a unique solution $\varphi \in W^{k+2,p}(\Omega)$ satisfying the estimate

$$
\|\varphi\|_{W^{k+2,p}(\Omega)} \leq C_1\|\text{div} \, v\|_{W^{k,p}(\Omega)} \leq C_1\|v\|_{W^{k+1,p}(\Omega)} \leq C_2\|h\|_{W^{k+1-1/p,p}(\Gamma)}.
$$

If we define $u = v - \nabla \varphi \in W^{k+1,p}(\Omega)$, we have $\text{div} \, u = \text{div} \, v - \Delta \varphi = 0$ in $\Omega$ and $u_T = v_T - (\nabla \varphi)_T = h_T = h$ on $\Gamma$ and

$$
\|u\|_{W^{k+1,p}(\Omega)} \leq \|v\|_{W^{k+1,p}(\Omega)} + \|\nabla \varphi\|_{W^{k+1,p}(\Omega)} \leq C\|h\|_{W^{k+1-1/p,p}(\Gamma)}.
$$

Next proposition which seems to be new characterizes the simply connectedness of the domain $\Omega$.

**Proposition 2.4.** Let $1 < p < \infty$. Then the domain $\Omega$ is simply connected, that is to say, the first Betti number is equal to zero if and only if there exists a constant $C = C(p, \Omega) > 0$ such that for all $u \in W^{1,p}(\Omega, \mathbb{R}^3)$,

$$
\|\nabla u\|_{L^p(\Omega)} \leq C(\|\text{curl} \, u\|_{L^p(\Omega)} + \|\text{div} \, u\|_{L^p(\Omega)} + \|\nu \cdot u\|_{W^{1-1/p,p}(\Gamma)}).
$$

Moreover in this case, the following estimate holds.

$$
\|u\|_{L^p(\Omega)} \leq C(\|\text{curl} \, u\|_{L^p(\Omega)} + \|\text{div} \, u\|_{L^p(\Omega)} + \|\nu \cdot u\|_{W^{1-1/p,p}(\Gamma)}).
$$

Proof. According to [11, Theorem 3.2], the first Betti number is equal to zero if and only if there exists a constant $C = C(p, \Omega) > 0$ such that for all $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ satisfying $\nu \cdot u = 0$ on $\Gamma$,

$$
\|\nabla u\|_{L^p(\Omega)} \leq C(\|\text{curl} \, u\|_{L^p(\Omega)} + \|\text{div} \, u\|_{L^p(\Omega)}).
$$

From this result, it is clear that (2.1) implies the simply connectedness. Thus it suffices to prove the necessity. For all $u \in W^{1,p}(\Omega, \mathbb{R}^3)$, it follows from Lemma 2.2 that there exists $v \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that $\nu \cdot v = \nu \cdot u$ on $\Gamma$, $\text{curl} \, v = 0$ in $\Omega$, and

$$
\|v\|_{W^{1,p}(\Omega)} \leq C\|\nu \cdot u\|_{W^{1-1/p,p}(\Gamma)}.
$$

If we put $w = u - v \in W^{1,p}(\Omega, \mathbb{R}^3)$, then $\nu \cdot w = 0$ on $\Gamma$. Therefore from (2.3), using (2.4) we have

$$
\|\nabla w\|_{L^p(\Omega)} \leq C(\|\text{curl} \, w\|_{L^p(\Omega)} + \|\text{div} \, w\|_{L^p(\Omega)}) \leq C(\|\text{curl} \, u\|_{L^p(\Omega)} + \|\text{curl} \, v\|_{L^p(\Omega)} + \|\text{div} \, u\|_{L^p(\Omega)} + \|\nu \cdot u\|_{W^{1-1/p,p}(\Gamma)}).
$$
From (2.4), we see that
\[ \| \nabla u \|_{L^p(\Omega)} \leq C(\| \text{curl} u \|_{L^p(\Omega)} + \| \text{div} u \|_{L^p(\Omega)} + \| \nu \cdot u \|_{W^{1-1/p,p}(\Gamma)}). \]
Assume that (2.2) is fault. From (2.1) we have
\[ \| u \|_{W^{1,p}(\Omega)} \leq C(\| u \|_{L^p(\Omega)} + \| \nabla u \|_{L^p(\Omega)} + \| \text{div} u \|_{L^p(\Omega)} + \| \nu \cdot u \|_{W^{1-1/p,p}(\Gamma)}). \] (2.5)
If (2.2) is fault, then there exists a sequence \( \{ u_j \} \subset W^{1,p}(\Omega, \mathbb{R}^3) \) such that
\[ \| u_j \|_{L^p(\Omega)} = 1, \text{ div } u_j \to 0 \text{ in } L^p(\Omega), \text{ curl } u_j \to 0 \text{ in } L^p(\Omega, \mathbb{R}^3) \text{ and } \nu \cdot u_j \to 0 \text{ in } W^{1-1/p,p}(\Gamma). \] From (2.5), passing to a subsequence, we may assume that \( u_j \to u \) weakly in \( W^{1,p}(\Omega, \mathbb{R}^3) \) and strongly in \( L^p(\Omega, \mathbb{R}^3). \) Thus we get \( \| u \|_{L^p(\Omega)} = 1, \text{ curl } u = 0, \text{ div } u = 0 \) and \( \nu \cdot u = 0 \) on \( \Gamma. \) Since \( \Omega \) is simply connected, there exists a function \( \phi \in W^{2,p}(\Omega) \) such that \( u = \nabla \phi \in \Omega. \) The function \( \phi \) satisfies the equation
\[ \begin{cases} \Delta \phi = \text{div } u = 0 & \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} = \nu \cdot u = 0 & \text{on } \Gamma. \end{cases} \]
Thus \( \phi = \text{constant}, \) so \( u = \nabla \phi = 0. \) This is a contradiction to \( \| u \|_{L^p(\Omega)} = 1. \)

The following two propositions follow from \([1, \text{ Corollary 3.5 and Corollary 5.3}]. \) In the particular case \( p = 2, \) see \([7]. \)

**Proposition 2.5.** Let \( k \geq 0 \) be an integer and \( \Omega \) satisfies (O1)-(O2) with \( r \geq k + 1. \) If \( u \in L^p(\Omega, \mathbb{R}^3) \) satisfies \( \text{div } u \in W^{k,p}(\Omega), \text{ curl } u \in W^{k,p}(\Omega, \mathbb{R}^3) \) and \( \nu \cdot u \in W^{k+1-1/p,p}(\Gamma), \) then we have \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3) \) and there exists a constant \( C = C(p, k, \Omega) \) such that
\[ \| u \|_{W^{k+1,p}(\Omega)} \leq C(\| u \|_{L^p(\Omega)} + \| \text{div} u \|_{W^{k,p}(\Omega)} + \| \text{curl} u \|_{W^{k,p}(\Omega)}) + \| \nu \cdot u \|_{W^{k+1-1/p,p}(\Gamma)}. \]

**Proposition 2.6.** Let \( k \geq 0 \) be an integer and \( \Omega \) satisfies (O1)-(O2) with \( r \geq k + 1. \) If \( u \in L^p(\Omega, \mathbb{R}^3) \) satisfies \( \text{div } u \in W^{k,p}(\Omega), \text{ curl } u \in W^{k,p}(\Omega, \mathbb{R}^3) \) and \( \nu \times u \in W^{k+1-1/p,p}(\Gamma, \mathbb{R}^3), \) then it follows that \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3) \) and there exists a constant \( C = C(p, k, \Omega) \) such that
\[ \| u \|_{W^{k+1,p}(\Omega)} \leq C(\| u \|_{L^p(\Omega)} + \| \text{div} u \|_{W^{k,p}(\Omega)} + \| \text{curl} u \|_{W^{k,p}(\Omega)} + \| \nu \times u \|_{W^{k+1-1/p,p}(\Gamma)}). \]

**Proposition 2.7.** Assume that \( \Omega \) is simply connected and let \( k \geq 0 \) be an integer. Then there exists a constant \( C \) such that for every \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3), \)
\[ \| u \|_{L^p(\Omega)} \leq C(\| \text{curl} u \|_{W^{k,p}(\Omega)} + \| \text{div} u \|_{W^{k,p}(\Omega)} + \| \nu \cdot u \|_{W^{k+1-1/p,p}(\Gamma)}). \]
Thus there exists a sequence \( \{ u_j \} \subset W^{k,1,p}(\Omega, \mathbb{R}^3) \) such that \( \| u_j \|_{L^p(\Omega)} = 1 \), and curl \( u_j \to 0 \) in \( W^{k,p}(\Omega, \mathbb{R}^3) \), div \( u_j \to 0 \) in \( W^{k,p}(\Omega) \) and \( \nu \cdot u_j \to 0 \) in \( W^{k+1-1/p,p}(\Gamma) \). By Proposition 2.5, we see that \( \{ u_j \} \) is bounded in \( W^{k+1,p}(\Omega, \mathbb{R}^3) \). Passing to a subsequence, we may assume that \( u_j \to u \) weakly in \( W^{k+1,p}(\Omega, \mathbb{R}^3) \) and strongly in \( L^p(\Omega, \mathbb{R}^3) \). Thus \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3) \) satisfies that curl \( u = 0 \), div \( u = 0 \) in \( \Omega \) and \( \nu \cdot u = 0 \) on \( \Gamma \), and \( \| u \|_{L^p(\Omega)} = 1 \). Thus \( u \in \mathbb{H}^p_1(\Omega) \). Since \( \Omega \) is simply connected, \( \mathbb{H}^p_1(\Omega) = \{ 0 \} \). Therefore \( u = 0 \) in \( \Omega \). This contradicts to \( \| u \|_{L^p(\Omega)} = 1 \). □

Combined Proposition 2.5 with Proposition 2.7, we easily get the following.

**Corollary 2.8.** Assume that \( \Omega \) is simply connected and let \( k \geq 0 \) be an integer. Then there exists a constant \( C \) such that for every \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3) \),

\[
\| u \|_{W^{k+1,p}(\Omega)} \leq C(\| \text{curl} \ u \|_{W^{k,p}(\Omega)} + \| \text{div} \ u \|_{W^{k,p}(\Omega)} + \| \nu \cdot u \|_{W^{k+1-1/p,p}(\Gamma)}) .
\]

Next proposition extends the result of [11, Theorem 3.1].

**Proposition 2.9.** Let \( 1 < p < \infty \). Then the domain \( \Omega \) has no holes, that is to say, the second Betti number is equal to zero if and only if there exists a constant \( C = C(p, \Omega) > 0 \) such that for all \( u \in W^{1,p}(\Omega, \mathbb{R}^3) \),

\[
\| \nabla u \|_{L^p(\Omega)} \leq C(\| \text{curl} \ u \|_{L^p(\Omega)} + \| \text{div} \ u \|_{L^p(\Omega)} + \| \nu \times u \|_{W^{1-1/p,p}(\Gamma)}) . \tag{2.6}
\]

Moreover in this case, the following estimate holds.

\[
\| u \|_{L^p(\Omega)} \leq C(\| \text{curl} \ u \|_{L^p(\Omega)} + \| \text{div} \ u \|_{L^p(\Omega)} + \| \nu \times u \|_{W^{1-1/p,p}(\Gamma)}) . \tag{2.7}
\]

**Proof.** According to [11, Theorem 3.1], the second Betti number is equal to zero if and only if there exists a constant \( C = C(p, \Omega) > 0 \) such that for all \( u \in W^{1,p}(\Omega, \mathbb{R}^3) \) satisfying \( \nu \times u = 0 \) on \( \Gamma \),

\[
\| \nabla u \|_{L^p(\Omega)} \leq C(\| \text{curl} \ u \|_{L^p(\Omega)} + \| \text{div} \ u \|_{L^p(\Omega)}) . \tag{2.8}
\]

From this result, it is clear that (2.6) implies that \( \Omega \) has no holes. Thus it suffices to prove the necessity. For all \( u \in W^{1,p}(\Omega, \mathbb{R}^3) \), it follows from Lemma 2.3 that there exists \( v \in W^{1,p}(\Omega, \mathbb{R}^3) \) such that \( \nu \times v = \nu \times u \) on \( \Gamma \), div \( v = 0 \) in \( \Omega \), and

\[
\| v \|_{W^{1,p}(\Omega)} \leq C\| \nu \times u \|_{W^{1-1/p,p}(\Gamma)}. \tag{2.9}
\]

If we put \( w = u - v \in W^{1,p}(\Omega, \mathbb{R}^3) \), then \( \nu \times w = 0 \) on \( \Gamma \). Therefore from (2.8), using (2.9) we have

\[
\| \nabla w \|_{L^p(\Omega)} \leq C(\| \text{curl} \ w \|_{L^p(\Omega)} + \| \text{div} \ w \|_{L^p(\Omega)}) \\
\leq C(\| \text{curl} \ u \|_{L^p(\Omega)} + \| \text{curl} \ v \|_{L^p(\Omega)} + \| \text{div} \ u \|_{L^p(\Omega)} + \| \text{div} \ v \|_{L^p(\Omega)}) \\
\leq C(\| \text{curl} \ u \|_{L^p(\Omega)} + \| \text{div} \ u \|_{L^p(\Omega)} + \| \nu \times u \|_{W^{1-1/p,p}(\Gamma)}). \]

Thus we have
\[ \| \nabla u \|_{L^p(\Omega)} \leq C(\| \text{curl} \, u \|_{L^p(\Omega)} + \| \text{div} \, u \|_{L^p(\Omega)} + \| \nu \times u \|_{W^{1-1/p,p}(\Gamma)}). \]

Assume that (2.7) is fault. From (2.6) we have
\[ \| u \|_{W^{1,p}(\Omega)} \leq C(\| \text{curl} \, u \|_{L^p(\Omega)} + \| \text{div} \, u \|_{L^p(\Omega)} + \| \nu \times u \|_{W^{1-1/p,p}(\Gamma)). \quad (2.10) \]

If (2.7) is fault, then there exists a sequence \( \{ u_j \} \subset W^{1,p}(\Omega, \mathbb{R}^3) \) such that \( \| u_j \|_{L^p(\Omega)} = 1 \), \( \text{curl} \, u_j \to 0 \) in \( L^p(\Omega) \), \( \text{div} \, u_j \to 0 \) in \( L^p(\Omega, \mathbb{R}^3) \) and \( \nu \cdot u_j \to 0 \) in \( W^{1-1/p,p}(\Gamma) \). From (2.10), passing to a subsequence, we may assume that \( u_j \to u \) weakly in \( W^{1,p}(\Omega, \mathbb{R}^3) \) and strongly in \( L^p(\Omega, \mathbb{R}^3) \). Thus we get \( \| u \|_{L^p(\Omega)} = 1 \), \( \text{curl} \, u = 0 \), \( \text{div} \, u = 0 \) and \( \nu \times u = 0 \) on \( \Gamma \). Since \( \Omega \) has no holes, we see that \( \mathbb{H}^p(\Omega) = \{ 0 \} \). Since \( u \in \mathbb{H}^p(\Omega) = \{ 0 \} \), this is a contradiction to \( \| u \|_{L^p(\Omega)} = 1 \).

**Proposition 2.10.** Assume that \( \Omega \) has no holes and let \( k \geq 0 \) be an integer. Then there exists a constant \( C \) such that for every \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3) \),
\[ \| u \|_{L^p(\Omega)} \leq C(\| \text{curl} \, u \|_{W^{k,p}(\Omega)} + \| \text{div} \, u \|_{W^{k,p}(\Omega)} + \| \nu \times u \|_{W^{k+1-1/p,p}(\Gamma)). \]

**Proof.** Assume that the conclusion is false. Then there exists a sequence \( \{ u_j \} \subset W^{k+1,p}(\Omega, \mathbb{R}^3) \) such that \( \| u_j \|_{L^p(\Omega)} = 1 \), \( \text{curl} \, u_j \to 0 \) in \( W^{k,p}(\Omega, \mathbb{R}^3) \), \( \text{div} \, u_j \to 0 \) in \( W^{k,p}(\Omega) \) and \( \nu \times u_j \to 0 \) in \( W^{k+1-1/p,p}(\Gamma, \mathbb{R}^3) \). By Proposition 2.6, we see that \( \{ u_j \} \) is bounded in \( W^{k+1,p}(\Omega, \mathbb{R}^3) \). Passing to a subsequence, we may assume that \( u_j \to u \) weakly in \( W^{k+1,p}(\Omega, \mathbb{R}^3) \) and strongly in \( L^p(\Omega, \mathbb{R}^3) \). Thus \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3) \) satisfies that \( \text{curl} \, u = 0 \), \( \text{div} \, u = 0 \) in \( \Omega \) and \( \nu \times u = 0 \) on \( \Gamma \), and \( \| u \|_{L^p(\Omega)} = 1 \). Thus \( u \in \mathbb{H}^p(\Omega) \). Since \( \Omega \) has no holes, \( \mathbb{H}^p(\Omega) = \{ 0 \} \). Therefore \( u = 0 \) in \( \Omega \). This contradicts to \( \| u \|_{L^p(\Omega)} = 1 \). \( \Box \)

Combined Proposition 2.6 with Proposition 2.10, we easily get the following.

**Corollary 2.11.** Assume that \( \Omega \) has no holes and let \( k \geq 0 \) be an integer. Then there exists a constant \( C \) such that for any \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3) \),
\[ \| u \|_{W^{k+1,p}(\Omega)} \leq C(\| \text{curl} \, u \|_{W^{k,p}(\Omega)} + \| \text{div} \, u \|_{W^{k,p}(\Omega)} + \| \nu \times u \|_{W^{k+1-1/p,p}(\Gamma)). \]

### 3 The div-curl system

In this section, we shall give some sufficient conditions in order to solve the div-curl system (1.1) or (1.2) and then we obtain the estimates of solutions by the given data.
Lemma 3.1. Assume that $\Omega \subset \mathbb{R}^3$ satisfies (O1)-(O2). Then the set $\text{curl} W^{1,p}(\Omega, \mathbb{R}^3)$ is closed subspace in $L^p(\Omega, \mathbb{R}^3)$ and
\[
\text{curl} W^{1,p}(\Omega, \mathbb{R}^3) = \{ u \in L^p(\Omega, \mathbb{R}^3); \text{div} u = 0 \text{ in } \Omega, \quad \langle \nu \cdot u, 1 \rangle_{\Gamma_j} = 0 \text{ for } j = 1, 2, \ldots, m+1 \}
\]
where $\langle \nu \cdot u, 1 \rangle_{\Gamma_j}$ is the duality of $W^{-1/p,p}(\Gamma) \times W^{1/p',p}(\Gamma)$ which is well defined from Lemma 2.1.

Proof. Let $v \in W^{1,p}(\Omega, \mathbb{R}^3)$. The equation
\[
\begin{cases}
\Delta \phi = \text{div} v & \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} = \nu \cdot v & \text{on } \partial \Omega
\end{cases}
\]
has a unique (up to an additive constant) solution $\phi \in W^{2,p}(\Omega)$. Then if we define $w = v - \nabla \phi \in W^{1,p}(\Omega, \mathbb{R}^3)$, we see that $\text{div} w = 0$ in $\Omega$, $\nu \cdot w = 0$ on $\partial \Omega$ and $\text{curl} w = \text{curl} v$ in $\Omega$. Thus we see that $\text{curl} W^{1,p}(\Omega, \mathbb{R}^3) = \text{curl} W^{1,p}_n(\Omega, \text{div} 0)$ where
\[
W^{1,p}_n(\Omega, \text{div} 0) = \{ w \in W^{1,p}(\Omega, \mathbb{R}^3); \text{div} w = 0 \text{ in } \Omega, \nu \cdot w = 0 \text{ on } \Gamma \}.
\]
We can easily see that $W^{1,p}_n(\Omega, \text{div} 0)$ is a Banach space with respect to the norm $\| \text{curl} w \|_{L^p(\Omega)} + \| w \|_{L^p(\Omega)}$ which is equivalent to $W^{1,p}$ norm. Thus we can apply the following Peetre lemma.

Lemma 3.2. Let $E_0, E_1, E_2$ be Banach spaces and $A_1 : E_0 \to E_2$ be a continuous linear operator, and $A_2 : E_0 \to E_2$ be a continuous and compact linear operator satisfying
\[
\| x \|_{E_0} \leq \| A_1 x \|_{E_1} + \| A_2 x \|_{E_2} \text{ for all } x \in E_0.
\]
Then $\text{Ker} A_1$ is of finite dimensional subspace and $\text{Im} A_1$ is closed subspace in $E_1$.

If we set $E_0 = W^{1,p}_n(\Omega, \text{div} 0)$, $E_1 = E_2 = L^p(\Omega, \mathbb{R}^3)$ and $A_1 = \text{curl}, A_2 = I_d$ where $I_d$ is the identity operator, it is easily seen that the above lemma is applicable. Thus we can see that
\[
\text{Im} A_1 = \text{curl} W^{1,p}_n(\Omega, \text{div} 0) = \text{curl} W^{1,p}(\Omega, \mathbb{R}^3)
\]
is a closed subspace in $L^p(\Omega, \mathbb{R}^3)$. It follow from [1, Lemma 4.1] that $u \in L^p(\Omega, \mathbb{R}^3)$ satisfies $\text{div} u = 0$ in $\Omega$ and $\langle \nu \cdot u, 1 \rangle_{\Gamma_j} = 0$ for $j = 1, 2, \ldots, m+1$ if and only if there exists $v \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that $u = \text{curl} v$ in $\Omega$. This completes the proof of Lemma 3.1. 

We are ready to state the main theorems in this paper.
Theorem 3.3. Assume that $k \geq 0$ is an integer and $\Omega$ satisfies (O1)-(O2) with $r \geq k + 1$. Let $\mathbf{B} \in W^{k,p}(\Omega, \mathbb{R}^3)$, $f \in W^{k,p}(\Omega)$ and $g \in W^{k+1-1/p,p}(\Gamma)$ satisfy the following conditions
\[
\text{div } \mathbf{B} = 0 \text{ in } \Omega, \quad \langle \nu \cdot \mathbf{B}, 1 \rangle_{\Gamma_j} = 0 \quad \text{for } j = 1, \ldots, m + 1,
\] (3.1)
and the compatibility condition:
\[
\int_{\Omega} f \, dx = \int_{\Gamma} g \, ds.
\] (3.2)
Then the system (1.1) has a solution $\mathbf{u} \in W^{k+1,p}(\Omega, \mathbb{R}^3)$ and there exists a constant $C_1 = C_1(\Omega, k, p)$ such that
\[
\|\mathbf{u}\|_{W^{k+1,p}(\Omega)} \leq C_1(\|\mathbf{B}\|_{W^{k,p}(\Omega)} + \|f\|_{W^{k,p}(\Omega)} + \|\mathbf{u}\|_{L^p(\Omega)} + \|g\|_{W^{k+1-1/p,p}(\Gamma)}).
\] (3.3)
In particular, if $\Omega$ is simply connected, then the solution is unique and there exists a constant $C_2 = C_2(\Omega, k, p)$ such that
\[
\|\mathbf{u}\|_{W^{k+1,p}(\Omega)} \leq C_2(\|\mathbf{B}\|_{W^{k,p}(\Omega)} + \|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k+1-1/p,p}(\Gamma)}).
\] (3.4)
Proof. Since $\mathbf{B}$ satisfies (3.1), it follows from Lemma 3.1 that there exists $\mathbf{A} \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that $\mathbf{B} = \text{curl } \mathbf{A}$. We consider the equation
\[
\begin{cases}
\Delta \phi = f - \text{div } \mathbf{A} & \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} = g - \nu \cdot \mathbf{A} & \text{on } \Gamma.
\end{cases}
\] (3.5)
Using (3.2) and the divergence theorem, we can see that
\[
\int_{\Omega} (f - \text{div } \mathbf{A}) \, dx = \int_{\Gamma} (g - \nu \cdot \mathbf{A}) \, ds.
\]
Hence the problem (3.5) has a solution $\phi \in W^{2,p}(\Omega)$ which is unique up to an additive constant. If we define $\mathbf{u} = \mathbf{A} + \nabla \phi \in W^{1,p}(\Omega, \mathbb{R}^3)$, then $\mathbf{u}$ satisfies that $\text{curl } \mathbf{u} = \text{curl } \mathbf{A} = \mathbf{B} \in W^{1,p}(\Omega, \mathbb{R}^3)$, $\text{div } \mathbf{u} = \text{div } \mathbf{A} + \Delta \phi = f \in W^{k,p}(\Omega)$ and $\nu \cdot \mathbf{u} = \nu \cdot \mathbf{A} + \frac{\partial \phi}{\partial \nu} = g \in W^{k+1-1/p,p}(\Gamma)$.
Therefore from Proposition 2.5 we see that $\mathbf{u} \in W^{k+1,p}(\Omega, \mathbb{R}^3)$ and the estimate (3.3) holds.

When $\Omega$ is simply connected, if $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^3)$ satisfies that $\text{curl } \mathbf{v} = 0$, $\text{div } \mathbf{v} = 0$ in $\Omega$ and $\nu \cdot \mathbf{v} = 0$ on $\Gamma$, then $\mathbf{v} \in \mathbb{H}^p(\Omega)$. Since $\Omega$ is simply connected, $\mathbb{H}^p(\Omega) = \{0\}$. Thus we have $\mathbf{v} = 0$. From this, the uniqueness of solution follows. The estimate (3.4) follows from Corollary 2.8. \hfill \Box

Proposition 3.4. Let $\mathbf{u} \in L^p(\Omega, \mathbb{R}^3)$. Then $\mathbf{u}$ satisfies that $\text{div } \mathbf{u} = 0$ in $\Omega$, $\nu \cdot \mathbf{u} = 0$ on $\Gamma$ and $\langle \nu \cdot \mathbf{u}, 1 \rangle_{\Sigma_i} = 0$ for $i = 1, 2, \ldots, n$ if and only if there exists a vector potential $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that $\mathbf{u} = \text{curl } \mathbf{v}$, $\text{div } \mathbf{v} = 0$ in $\Omega$, $\nu \times \mathbf{v} = 0$ on $\Gamma$ and $\langle \nu \cdot \mathbf{v}, 1 \rangle_{\Gamma_j} = 0$ for $j = 1, 2, \ldots, m$. Moreover, $\mathbf{v}$ is unique and there exists a constant $C > 0$ such that
\[
\|\mathbf{v}\|_{W^{1,p}(\Omega)} \leq C\|\mathbf{u}\|_{L^p(\Omega)}.
\]
The proof was given in [1, Theorem 4.3].

**Theorem 3.5.** Assume that \( k \geq 0 \) is an integer and \( \Omega \) satisfies (O1)-(O2) with \( r \geq k + 1 \). Let \( B \in W^{k,p}(\Omega, \mathbb{R}^3) \), \( f \in W^{k,p}(\Omega) \) and \( h \in W^{k+1-1/p,p}(\Gamma, \mathbb{R}^3) \) satisfy the following conditions

\[
\begin{align*}
\text{div} \ B & = 0 \text{ in } \Omega, \quad \nu \cdot h = 0, \quad \nu \cdot B = \nu \cdot \text{curl} \ (h \times \nu) \text{ on } \Gamma, \\
& \text{(3.6)}
\end{align*}
\]

and there exists \( H \in W^{1,p}(\Omega, \mathbb{R}^3) \) such that

\[
\nu \times H = h \text{ on } \Gamma \text{ and } (\nu \cdot (\text{curl} \ H - B), 1)_{\Sigma_i} = 0 \text{ for } i = 1, 2, \ldots, n. \quad (3.7)
\]

Then the system (1.2) has a solution \( u \in W^{k+1,p}(\Omega, \mathbb{R}^3) \) and there exists a constant \( C_3 = C_3(\Omega, k, p) \) such that

\[
\|u\|_{W^{k+1,p}(\Omega)} \leq C_3(\|B\|_{W^{k,p}(\Omega)} + \|f\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)} + \|h\|_{W^{k+1-1/p,p}(\Gamma)}). \quad (3.8)
\]

In particular, if \( \Omega \) has no holes, then the solution is unique and there exists a constant \( C_4 = C_4(\Omega, k, p) \) such that

\[
\|u\|_{W^{k+1,p}(\Omega)} \leq C_4(\|B\|_{W^{k,p}(\Omega)} + \|f\|_{W^{k,p}(\Omega)} + \|h\|_{W^{k+1-1/p,p}(\Gamma)}). \quad (3.9)
\]

**Proof.** First the equation

\[
\begin{align*}
\Delta \phi &= f - \text{div} \ H \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \Gamma
\end{align*}
\]

has a unique solution \( \phi \in W^{2,p}(\Omega) \). Then we put \( u = v + H + \nabla \phi \) and rewrite the equation (1.2) into the system with respect to \( v \). Since

\[
\begin{align*}
\text{curl} \ v &= \text{curl} \ u - \text{curl} \ H = B - \text{curl} \ H \text{ in } \Omega, \\
\text{div} \ v &= \text{div} \ u - \text{div} \ H - \Delta \phi = f - \text{div} \ H - \Delta \phi = 0 \text{ in } \Omega, \\
\nu \times v &= \nu \times u - \nu \times H - \nu \times \nabla \phi = h - h = 0 \text{ on } \Gamma,
\end{align*}
\]

we have the new system

\[
\text{curl} \ v = B - \text{curl} \ H, \quad \text{div} \ v = 0 \text{ in } \Omega, \quad \nu \times v = 0 \text{ on } \Gamma. \quad (3.10)
\]

From (3.6), we see that \( \text{div} \ (B - \text{curl} \ H) = \text{div} \ B = 0 \) in \( \Omega \), and on \( \Gamma \),

\[
\begin{align*}
\nu \cdot (B - \text{curl} \ H) &= \nu \cdot B - \nu \cdot \text{curl} \ H \\
&= \nu \cdot B - \nu \cdot \text{curl} \ H_T \\
&= \nu \cdot B - \nu \cdot \text{curl} \ ((\nu \times H) \times \nu) \\
&= \nu \cdot B - \nu \cdot \text{curl} \ (h \times \nu) = 0.
\end{align*}
\]
Here we used the property $\nu \cdot \text{curl} \mathbf{H} = \nu \cdot \text{curl} \mathbf{H}_T$ (cf. Monneau [9]). Moreover, from (3.7), we have $\langle \nu \cdot (\mathbf{B} - \text{curl} \mathbf{H}), 1 \rangle_{\Sigma_i} = 0$ for $i = 1, 2, \ldots, n$. Thus it follows from Proposition 3.4 that there exists $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^3)$ such that $\text{curl} \mathbf{v} = \mathbf{B} - \text{curl} \mathbf{H}$, $\text{div} \mathbf{v} = 0$ in $\Omega$, $\nu \times \mathbf{v} = 0$ on $\partial \Omega$ and $\langle \nu \cdot \mathbf{v}, 1 \rangle_{\Gamma_j} = 0$ for $j = 1, 2, \ldots, m$. Thus $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^3)$ is a solution of (3.10). Therefore $\mathbf{u} = \mathbf{v} + \mathbf{H} + \nabla \phi \in W^{1,p}(\Omega, \mathbb{R}^3)$ is a solution of (1.2). From the system (1.2) and Proposition 2.6, we have $\mathbf{u} \in W^{k,p}(\Omega, \mathbb{R}^3)$ and the estimate (3.8) holds. When $\Omega$ has no holes, if $\text{curl} \mathbf{w} = 0$, $\text{div} \mathbf{w} = 0$ in $\Omega$ and $\nu \times \mathbf{w} = 0$ on $\Gamma$, then $\mathbf{w} \in H^p_2(\Omega)$. Since $\Omega$ has no holes, we see that $H^p_2(\Omega) = \{0\}$. Hence we see that $\mathbf{w} = 0$. The estimate (3.9) follows from Corollary 2.11.

References


[5] Bolik, J. and Wahl, W., Estimating $\nabla \mathbf{u}$ in terms of $\text{div} \mathbf{u}$, $\text{curl} \mathbf{u}$, either $(\nu, \mathbf{u})$ or $\nu \times \mathbf{u}$ and the topology, Math. Methods Appl. Sci., 20, (1997), 734-744.


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