On the Twisted Tangent Numbers and Polynomials of Higher Order Associated with Multiple Twisted Tangent Zeta Function

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Abstract

In this paper, we introduce the twisted Tangent polynomials $T_{n,\omega}^{(k)}(x)$ of higher order. Finally we construct multiple twisted Tangent zeta function which interpolates the twisted Tangent numbers of higher order at negative integers.

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1 Introduction

Numerous properties of tangent number are known. Many mathematicians have studied in the area of the analogues of the Bernoulli numbers, Euler numbers, and Genocchi numbers (see [1-6]). In [4], we introduce the twisted Tangent numbers $T_{n,\omega}(x)$ and investigate their properties. Our aim in this paper is to define the twisted Tangent polynomials $T_{n,\omega}^{(k)}(x)$ of higher order $k$. We also derive the existence of a specific interpolation function which interpolate twisted Tangent polynomials $T_{n,\omega}^{(k)}(x)$ of higher order $k$ at
negative integers. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p}}$ so that $q = \exp(x \log q)$ for $|x|_p \leq 1$.

2 Twisted Tangent polynomials of higher order

In this section, our goal is to define the twisted Tangent numbers and polynomials of higher order. Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order $p^N$. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto \omega^x$ (see [4]). Now, using multiple of $p$-adic integral on $\mathbb{Z}_p$, we introduce the twisted Tangent polynomials of higher order $T_{n, \omega}^{(k)}(x)$: For $k \in \mathbb{N}$, we define

$$\sum_{n=0}^{\infty} T_{n, \omega}^{(k)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1 + \cdots + x_k} e^{(x + 2x_1 + \cdots + 2x_k)t} d\mu_1(x_1) \cdots d\mu_1(x_k).$$

By using Taylor series of $e^{(x + 2x_1 + \cdots + 2x_k)t}$ in the above equation, we obtain

$$\sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1 + \cdots + x_k} (x + 2x_1 + \cdots + 2x_k)^n d\mu_1(x_1) \cdots d\mu_1(x_k) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} T_{n, \omega}^{(k)}(x) \frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.
Theorem 2.1 For positive integers \( n \) and \( k \), one has

\[
T_{n,\omega}^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1 + \cdots + x_k} (x + 2x_1 + \cdots + 2x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).
\]

Observe that for \( x = 0 \), the Theorem 2.1 reduces to Corollary 2.2.

Corollary 2.2 For positive integers \( n \) and \( k \), one has

\[
T_{n,\omega}^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1 + \cdots + x_k} (2x_1 + \cdots + 2x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).
\]

The twisted Tangent polynomials of higher order, \( T_{n,\omega}^{(k)}(x) \) are defined by means of the following generating function

\[
F_{\omega}^{(k)}(x, t) = \left( \frac{2}{\omega e^{2t} + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} T_{n,\omega}^{(k)}(x) \frac{t^n}{n!}.
\]  

(2.1)

It follows from (2.1) that

\[
\lim_{\omega \to 1} F_{\omega}^{(k)}(x, t) = \left( \frac{2e^t}{e^{2t} + 1} \right)^k e^{xt}.
\]

This gives a generating function of the Tangent polynomials of higher order. Thus we have the following limit relationship:

\[
\lim_{\omega \to 1} T_{n,\omega}^{(k)}(x) = T_{n}^{(k)}(x),
\]

which yields the Tangent polynomials of higher order as a limit as \( \omega \) approaches 1 (see [2]). By using (2.1), the twisted Tangent numbers of higher order, \( T_{n,\omega}^{(k)} \) are defined by the following generating function

\[
\left( \frac{2}{\omega e^{2t} + 1} \right)^k = \sum_{n=0}^{\infty} T_{n,\omega}^{(k)} \frac{t^n}{n!}, \quad |2t + \log \omega| < \pi.
\]  

(2.2)

When \( k = 1 \), above (2.1) and (2.2) will become the corresponding definitions of the twisted Tangent polynomials \( T_{n,\omega}(x) \) and the twisted Tangent numbers \( T_{n,\omega} \) (see [3]). By using binomial expansion in Theorem 2.1, we obtain

\[
T_{n,\omega}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1 + \cdots + x_k} (2x_1 + \cdots + 2x_k)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).
\]

By Corollary 2.2, we arrive at the following theorem.
Theorem 2.3 For positive integers $n, k$, we have
\[ T_{n, \omega}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} T_{l, \omega}^{(k)}. \]

Next, we define distribution relation of the twisted Tangent polynomials of higher order as follows: For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, we obtain
\[
\sum_{n=0}^{\infty} T_{n, \omega}^{(k)}(x) \frac{t^n}{n!} = \left( \frac{2}{\omega e^{2t} + 1} \right) \left( \frac{2}{\omega e^{2t} + 1} \right) \cdots \left( \frac{2}{\omega e^{2t} + 1} \right) e^{xt}
\]
\[
= \left( \frac{2}{\omega^{m} e^{2mt} + 1} \right)^k \sum_{a_1, \ldots, a_k = 0}^{m-1} (-\omega)^{a_1 + \cdots + a_k} e^\left( \frac{2a_1 + \cdots + 2a_k + x}{m} \right) (mt)^n.
\]

From the above, we have
\[
\sum_{n=0}^{\infty} T_{n, \omega}^{(k)}(x) \frac{t^n}{n!} = \sum_{a_1, \ldots, a_k = 0}^{m-1} (-\omega)^{a_1 + \cdots + a_k} \sum_{n=0}^{\infty} T_{n, \omega}^{(k)} \left( \frac{2a_1 + \cdots + 2a_k + x}{m} \right) (mt)^n / n!.
\]

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 2.4 (Distribution relation of the twisted Tangent polynomials of higher order). For $m \in \mathbb{N}$ with $m \equiv 1 \pmod{2}$, one has
\[ T_{n, \omega}^{(k)}(x) = m^n \sum_{a_1, \ldots, a_k = 0}^{m-1} (-\omega)^{a_1 + \cdots + a_k} T_{n, \omega}^{(k)} \left( \frac{2a_1 + \cdots + 2a_k + x}{m} \right). \]

By (2.1), we have
\[
\sum_{n=0}^{\infty} T_{n, \omega}^{(k)}(x) \frac{t^n}{n!} = 2^k \sum_{a_1, \ldots, a_k = 0}^{m-1} (-1)^{a_1 + \cdots + a_k} \omega^{a_1 + \cdots + a_k} e^{(2a_1 + \cdots + 2a_k + x)t}
\]
\[
= 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m \omega^m (2m+x)^t.
\]

From the above, we have
\[
\sum_{n=0}^{\infty} T_{n, \omega}^{(k)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( 2^k \sum_{a_1, \ldots, a_k = 0}^{m-1} (-\omega)^{a_1 + \cdots + a_k} (x + 2a_1 + \cdots + 2a_k)^n \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m \omega^m (2m+x)^n \right) \frac{t^n}{n!}.
\]

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.
Theorem 2.5 For positive integers \( n,k \), one has

\[
T^{(k)}_{n,\omega}(x) = 2^k \sum_{a_1,\ldots,a_k=0}^{\infty} (-1)^{a_1+\cdots+a_k} \omega^{a_1+\cdots+a_k} (2a_1 + \cdots + 2a_k + x)^n \\
= 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m \omega^m (2m + x)^n.
\]  

(2.4)

By definition of the twisted Tangent polynomials of higher order, we have the following addition theorem.

Theorem 2.6 (Addition theorem of the twisted Tangent polynomials of higher order). For \( k \in \mathbb{N} \), one has

\[
T^{(k)}_{n,\omega}(x+y) = \sum_{l=0}^{n} \binom{n}{l} T^{(k)}_{l,\omega}(x) y^{n-l}.
\]

3 Multiple Twisted Tangent zeta function

In this section, we define multiple twisted Tangent zeta function. This function interpolates the twisted Tangent numbers of higher order at negative integers. Let \( w \) be the \( p^{N} \)-th root of unity.

We define multiple twisted Tangent zeta function. This function interpolates the twisted Tangent numbers of higher order at negative integers. By using (2.1), we have

\[
F^{(k)}_{\omega}(x,t) = 2^k \sum_{a_1,\ldots,a_k=0}^{\infty} (-\omega)^{a_1+\cdots+a_k} e^{(2a_1+\cdots+2a_k+x)t} = \sum_{n=0}^{\infty} T^{(k)}_{n,\omega}(x) \frac{t^n}{n!}.
\]  

(3.1)

From these generating functions of the twisted Tangent polynomials of higher order, we construct Hurwitz’s type multiple twisted Tangent zeta function as follows:

Definition 3.1 For \( s, x \in \mathbb{C} \) with \( \Re(x) > 0 \), we define

\[
\zeta^{(k)}_{\omega}(s,x) = 2^k \sum_{a_1,\ldots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k} \omega^{a_1+\cdots+a_k}}{(2a_1+\cdots+2a_k+x)^s}.
\]  

(3.2)

By the \( l \)-th differentiation on the both side of (3.1) at \( t = 0 \), we obtain the following

\[
T^{(k)}_{n,\omega}(x) = \left( \frac{d}{dt} \right)_l F^{(k)}_{\omega}(x,t) \bigg|_{t=0} = 2^k \sum_{a_1,\ldots,a_k=0}^{\infty} (-\omega)^{a_1+\cdots+a_k} (2a_1+\cdots+2a_k+x)^l.
\]
From (3.1) and (3.2), we arrive at the following theorem.

**Theorem 3.2** For positive integer \( l \), one has

\[
\zeta^{(k)}_{\omega}(-l, x) = T^{(k)}_{l, \omega}(x).
\]

By (2.2), we have

\[
\sum_{n=0}^{\infty} T^{(k)}_{n, \omega} \frac{t^n}{n!} = \left( \frac{2}{\omega e^{2t} + 1} \right)^k = 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m \omega^m e^{(2m)t}.
\]

By using Taylor series of \( e^{(2m)t} \) in the above, we have

\[
\sum_{n=0}^{\infty} T^{(k)}_{n, \omega} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m \omega^m (2m)^n \right) \frac{t^n}{n!}.
\]

By comparing coefficients of \( \frac{t^n}{n!} \) in the above equation, we have

\[
T^{(k)}_{n, \omega} = 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m \omega^m (2m)^n. \tag{3.3}
\]

By using (3.3), we define multiple twisted Tangent zeta function as follows:

**Definition 3.3** For \( s \in \mathbb{C} \), we define

\[
\zeta_{\omega}^{(k)}(s) = 2^k \sum_{m=1}^{\infty} \binom{m+k-1}{m} \frac{(-1)^m \omega^m}{(2m)^s}. \tag{3.4}
\]

The function \( \zeta_{\omega}^{(k)}(s) \) interpolates the number \( T^{(k)}_{n, \omega} \) at negative integers. Substituting \( s = -n \) with \( n \in \mathbb{Z}_+ \) into (3.4), and using (3.3), we have the following theorem:

**Theorem 3.4** Let \( n \in \mathbb{Z}_+ \), one has

\[
\zeta_{\omega}^{(k)}(-n) = E^{(k)}_{n, \omega}.
\]

**References**


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