Some Results on Fixed Points of

Asymptotically Regular Mappings

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Abstract

In this paper we extend some results of fixed points for asymptotically regular mappings on a complete 2-metric space. We generalize some fixed point theorem of ‘Slobodan C. Nesic’ in the context of 2-metric space

Keywords: Fixed Point, Asymptotically Regular Mapping, 2-metric space

1 Introduction

The concept of asymptotically regular at a point in a space was first introduced by Browder and Petryshyn [1]

2 Preliminaries

Definition 2.1: A 2-metric space is a set X with a real valued non-negative function is defined on $X \times X \times X$ such that
(1) for all \( x, y \in X \), \( (x \neq y) \), there exists a point \( z \in X \) such that \( \sigma(x, y, z) \neq 0 \)

(2) \( \sigma(x, y, z) = 0 \) if at least two of points \( x, y, z \) coincide.

(3) \( \sigma(x, y, z) = \sigma(x, z, y) = \sigma(y, z, x) = \sigma(y, x, z) \)

(4) \( \sigma(x, y, z) \leq \sigma(x, y, w) + \sigma(x, w, z) + \sigma(w, y, z) \)

The function \( \sigma \) is called 2-metric for the space and \((X, \sigma)\) is called a 2-metric space.

**Definition 2.2:** A mapping \( T: X \to X \) of a 2-metric space \((X, \sigma)\) into itself is said to be asymptotically regular at a point \( x \in X \) if

\[
\lim_{n \to \infty} (T^n x, T^{n+1} x, z) = 0, \quad (z \in X).
\]

### 3 Main Results

In the following theorem, we establish a unique fixed point in \( X \)

**Theorem 3.1**

Let \( (X, \sigma) \) be a complete 2-metric space and \( T: X \to X \) be a mapping such that the following condition is satisfied.

(i) \( \sigma(Tx, Ty, z) \leq p \sigma(x, y, z) + q[\sigma(x, Tx, z) + \sigma(y, Ty, z)] \)

\[
+ r[\sigma(x, Ty, z) + \sigma(y, Tx, z)] \\
+ F[\sigma(x, Tx, z) \cdot \sigma(y, Ty, z)]
\]

for all \( x, y, z \in X \), \( 0 \leq p, r, p + 2r < 1 \), \( q + r < 1 \).

If \( T \) is asymptotically regular at some point of \( X \), then \( T \) has a unique fixed point in \( X \).

**Proof:** We shall assume that \( T \) is asymptotically regular at a point \( x_0 \in X \) and consider the sequence \( \{T^n x_0\} \). Then

\[
\sigma(T^{m-1} x_0, T^n x_0, z) \leq p \sigma(T^{m-1} x_0, T^n x_0, z) \\
+ q[\sigma(T^{m-1} x_0, T^n x_0, z) + \sigma(T^n x_0, T^{m-1} x_0, z)] \\
+ r[\sigma(T^{m-1} x_0, T^{n-1} x_0, z) + \sigma(T^{n-1} x_0, T^{m-1} x_0, z)] \\
+ F[\sigma(T^{m-1} x_0, T^n x_0, z) \cdot \sigma(T^{n-1} x_0, T^m x_0, z)]
\]

\[
\leq p \left[ \sigma(T^{m-1} x_0, T^n x_0, z) + \sigma(T^n x_0, T^{m-1} x_0, z) \right] \\
+ q[\sigma(T^{m-1} x_0, T^n x_0, z) + \sigma(T^n x_0, T^{m-1} x_0, z)] \\
+ r[\sigma(T^{m-1} x_0, T^{n-1} x_0, z) + \sigma(T^{n-1} x_0, T^{m-1} x_0, z)] \\
+ F[\sigma(T^{m-1} x_0, T^n x_0, z) \cdot \sigma(T^{n-1} x_0, T^m x_0, z)]
\]
Some results on fixed points of asymptotically regular mappings

\[ \lim_{n \to \infty} T^n x_0 = n \]

\[ \leq p[\sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(T^n x_0, T^n x_0, z) + \sigma(T^n x_0, T^{n-1}x_0, z) + \sigma(T^n x_0, T^n x_0, 0)] \]

\[ + \sigma(T^n x_0, T^n x_0, 0)] \]

\[ + q[\sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(T^{n-1}x_0, T^n x_0, z)] \]

\[ + 2r \, \sigma(T^n x_0, T^n x_0, z) + r[\sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(T^{n-1}x_0, T^n x_0, z)] \]

\[ + F[\sigma(T^{n-1}x_0, T^n x_0, z) \cdot \sigma(T^{n-1}x_0, T^n x_0, z)] \]

\[ \text{i.e. } \sigma(T^n x_0, T^n x_0, z) \leq \frac{(p+q+r)}{1-p-2r} [\sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(T^{n-1}x_0, T^n x_0, z)] \]

\[ + \frac{1}{1-p-2r}F[\sigma(T^{n-1}x_0, T^n x_0, z) \cdot \sigma(T^{n-1}x_0, T^n x_0, z)] \]

Since \( T \) is asymptotically regular at \( x_0 \),
\[ \sigma(T^n x_0, T^n x_0, z) \to 0 \text{ as } m, n \to \infty, \quad z \in X \]

\[ \Sigma (T^{n-1}x_0, T^n x_0, T^n x_0) \to 0 \]

\[ \sigma(T^{n-1}x_0, T^n x_0, T^n x_0) \to 0, \quad \text{etc, as } m, n \to \infty \]

Hence \( \{T^n x_0\} \) is a Cauchy sequence.

Since \( (X, \sigma) \) is complete, there exist a point \( u \in X \) such that
\[ u = \lim_{n \to \infty} T^n x_0 \]

Suppose that \( u \) is not a fixed point of \( T \),
Then by (i), we obtain
\[ \sigma(u, Tu, z) \leq \sigma(u, T^n x_0, z) + \sigma(T^n x_0, Tu, z) \]

\[ \leq \sigma(u, T^n x_0, z) + p[\sigma(T^{n-1}x_0, u, z)] + q[\sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(u, Tu, z)] \]

\[ + r [\sigma(T^{n-1}x_0, Tu, z) + \sigma(u, T^n x_0, z)] \]

\[ + F[\sigma(T^{n-1}x_0, T^n x_0, z) \cdot \sigma(u, Tu, z)] \]

\[ \leq \sigma(u, T^n x_0, z) + p[\sigma(T^{n-1}x_0, u, z)] \]

\[ + q [\sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(u, Tu, z)] + r \, \sigma(u, T^n x_0, z) \]

\[ + r [\sigma(u, Tu, z) + \sigma(T^{n-1}x_0, u, z) + \sigma(T^{n-1}x_0, Tu, z)] \]

\[ + F \left[ \sigma(T^{n-1}x_0, T^n x_0, z) \cdot \sigma(u, Tu, z) \right] \]

\[ = (1+r) \, \sigma(u, T^n x_0, z) + (p+r) \, \sigma(u, T^{n-1}x_0, z) + q \, \sigma(T^{n-1}x_0, T^n x_0, z) \]

\[ + (q+r) \, \sigma(u, Tu, z) + F \left[ \sigma(T^{n-1}x_0, T^n x_0, z) \cdot \sigma(u, Tu, z) \right] \]

Taking the limit as \( n \to \infty \), we obtain \( \sigma(u, Tu, z) \leq (q+r) \, \sigma(u, Tu, z) \)
Which contradicts \( (q+r) < 1 \) unless \( u = Tu \),
Suppose \( T \) has second fixed point \( v \) in \( X \). Then by (i),
We obtain \( \sigma(u, v, z) \leq (p+2r) \, \sigma(u, v, z) \)
Since \((p + 2r) < 1\) it follows that \(u = v\).
Hence the fixed point is unique.

**Theorem 3.2**

Let \((X, \sigma)\) be a 2-metric space and \(T\) be a mapping of \(X\) into itself satisfying the condition
\[
\sigma(Tx, Ty, z) \leq p\sigma(x, y, z) + q[\sigma(x, Tx, z) + \sigma(y, Ty, z)] + r[\sigma(x, Ty, z) + \sigma(y, Tx, z)] + F[\sigma(x, Tx, z) \cdot \sigma(y, Ty, z)] \quad \text{for all } x, y, z \in X,
\]
\(0 \leq p, r, p + 2r < 1, \quad q + r < 1\).

If \(T\) is asymptotically regular at a point \(x \in X\) and sequence of iterates \(\{T^nx\}\) has a subsequence converging to a point \(z \in X\), then \(z\) is a unique fixed point of \(T\) and \(\{T^nx\}\) also converges to \(z\).

**Proof:**

Let \(T\) be asymptotically regular at \(x \in X\) and consider the sequence \(\{T^nx\}\), we shall assume that \(\lim_{k \to \infty} T_{nk}^k = z\) and \(Tz \neq z\).

Then by condition (i), we obtain, (for \(u \in X\))
\[
\sigma(z, Tz, u) \leq \sigma(z, T^nkx, u) + \sigma(T^nkx, T^{nk+1}x, u) + \sigma(T^{nk+1}x, Tz, u)
\]
\[
\leq \sigma(z, T^nkx, u) + \sigma(T^nkx, T^{nk+1}x, u) + p \sigma(T^nkx, z, u) + q[\sigma(T^nkx, T^{nk+1}x, u) + \sigma(z, Tz, u)] + r[\sigma(z, T^{nk+1}x, u) + \sigma(T^nkx, Tz, u)] + F[\sigma(T^nkx, T^{nk+1}x, u) \cdot \sigma(z, Tz, u)]
\]
\[
\leq \sigma(z, T^nkx, u) + \sigma(T^nkx, T^{nk+1}x, u) + p \sigma(T^nkx, z, u) + q[\sigma(T^nkx, T^{nk+1}x, u) + \sigma(z, Tz, u)] + r[\sigma(z, T^{nk+1}x, u) + \sigma(z, Tz, u) + \sigma(T^nkx, Tz, z)] + F[\sigma(T^nkx, T^{nk+1}x, u) \cdot \sigma(z, Tz, u)]
\]

Taking limit as \(k \to \infty\), we obtain
\[
\sigma(z, Tz, u) \leq (q + r) \sigma(z, Tz, u) \quad \text{which is contradicts } q + r < 1 \text{ unless } z = Tz.
\]

By Theorem 3.1, \(z\) is the unique fixed point. By using (i), we obtain
\[ \sigma(z, T^n x, u) = \sigma(Tz, T^n z, u) \leq \sigma(Tz, T^{n+1} x, u) + \sigma(T^n x, T^{n+1} x). \]

\[ \therefore \sigma(z, T^n x, u) \leq \sigma(Tz, T^{n+1} x, u) + \sigma(T^n x, u) \]

\[ \leq p \sigma(z, T^n x, u) + q(\sigma(z, Tz, u) + q\sigma(T^n x, T^{n+1} x, u)) + r[\sigma(z, T^n x, u) + \sigma(T^n x, Tz, u)] + F[\sigma(z, Tz, u) \cdot \sigma(T^n x, T^{n+1} x, u)] \]

\[ \leq p \sigma(z, T^n x, u) + q[\sigma(z, Tz, u) + \sigma(T^n x, T^{n+1} x, u)] + r[\sigma(z, T^n x, u) + \sigma(T^n x, T^{n+1} x, u)] + \sigma(z, Tz, u) + \sigma(T^n x, Tz, z) + \sigma(T^{n+1} x, T^n x, u) \]

This implies that
\[ \sigma(z, T^n x, u) \leq \frac{(1+q+r)}{(1-p-2r)} \sigma(T^n x, T^{n+1} x, u), \quad (x \in X) \]

Since \( p + 2r < 1 \), \( Tz = z \).

Since \( T \) is asymptotically regular, \( \lim_{n \to \infty} \sigma(z, T^n x, u) = 0 \).

This implies that \( \{T^n x\} \) converges to \( z \).

This completes the proof.

References


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