The Fluid of Couette and the Boundary Layer

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Abstract

In this paper, we find a solution for a temperature problem in a layer of fluid between two walls and also it is studied using a dynamical system.

Mathematics Subject Classification: 35K05

Keywords: Boundary layer, Incomplete Gamma function
1 Introduction

We can find something about thermal conductivity with some velocity distribution and fluid flow in [1, 2, 3]. In this paper, we study the temperature of a fluid near a solid wall, where the temperature will be obtained from a partial differential equation which will be resolved by the method of the asymptotic expansions with some boundary conditions and we solved the equation (2) that it is not easy to solve it.

2 Preliminary

Consider fluid flow inside two parallel walls of length $L$ and these are separated by a distance $h$. The temperature $T$ satisfies:

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{Pe} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \]  

(1)

where $Pe >> 1$ is the Peclet number.

The boundary conditions for the problem are

\[
T(0, y) = T_0 \quad 0 < y < h \\
T(x, 0) = T_w \quad 0 < x < 1 \\
T(x, h) = T_w \quad 0 < x < 1 \\
T(1, y) = T_w \quad 0 < y < h.
\]

Here, $T_0$ is the fluid inlet temperature which is constant and suppose that $T_0 > T_w$. Temperature profile at the exit is $T_w$.

3 Main Result

**Theorem 3.1.** The problem (1) has an approximated solution (9)

Proof. Replacing the small parameter $\frac{1}{Pe}$ by $\epsilon$ and the values of $u = y$ and $v = 0$ respectively in (1) and analyzing the boundary layer in one dimension

\[
y \frac{\partial T}{\partial x} = \epsilon \left( \frac{\partial^2 T}{\partial x^2} \right) \quad 0 < x < 1 \\
T(0) = T_0 \text{ and } T(1) = T_w
\]

we have the following general solution $T = A + \epsilon Be^{yx/\epsilon}/y$ where $A$ and $B$ are constants and using initial conditions and $e^{-y/\epsilon} \to 0$ we have $T = T_0 + (T_w - T_0)e^{-(1-x)y/\epsilon} + \ldots$
If the outer expansion is

\[ T^{\text{out}}(x, y, \varepsilon) = T^{\text{out}}_0(x, y) + \varepsilon T^{\text{out}}_1(x, y) + \varepsilon^2 T^{\text{out}}_2(x, y) + \ldots \]

we obtain

\[
O(1): \quad y \frac{\partial T^{\text{out}}_0(x, y)}{\partial x} = 0
\]

\[
O(\varepsilon^n): \quad y \frac{\partial T^{\text{out}}_n(x, y)}{\partial x} = \frac{\partial^2 T^{\text{out}}_n(x, y)}{\partial x^2} + \frac{\partial T^{\text{out}}_{n-1}(x, y)}{\partial y^2}
\]

for \( n \geq 1 \). And \( T^{\text{out}}_0(x, y) = T_0 \)

Suppose we have the transformation of the form \( s = y/\varepsilon^p \). We obtain that

\[
\frac{s}{\partial T}{\partial x} = \frac{1}{\varepsilon^{p-1}} \frac{\partial^2 T}{\partial x^2} + \frac{1}{\varepsilon^{3p-1}} \frac{\partial^2 T}{\partial s^2}.
\]

The left-hand side of the equation is \( O(1) \) and we require that \( p = 1/3 \) then the one-term inner expansion \( T^{\text{in}}_0(x, s) \) holds that

\[
\frac{s}{\partial T^{\text{in}}_0}{\partial x} = \frac{\partial^2 T^{\text{in}}_0}{\partial s^2}.
\] (2)

We will find the general solution by combination of variables.

Assume

\[ T^{\text{in}}_0(x, y) = T_0 \phi(\eta), \]

where

\[ \eta = \frac{s}{\delta(x)} \]

Now we obtain the derivatives with respect to \( x \) and \( s \)

\[ \frac{\partial T^{\text{in}}_0}{\partial x} = -T_0 \eta \frac{\partial \phi}{\partial \eta} \frac{\partial \delta(x)}{\partial x}, \]

and

\[ \frac{\partial^2 T^{\text{in}}_0}{\partial s^2} = T_0 \frac{1}{\delta^2(x)} \frac{\partial^2 \phi}{\partial \eta^2}. \]

Substituting these results in (2) leads the following differential equation

\[ \phi'' + \eta^2 \delta^2(x) \delta'(x) \phi' = 0. \] (3)
We suppose that
\[ \delta^2(x)\delta'(x) = 2 \quad (4) \]
so the equation (3) is
\[ \phi'' + 2\eta^2 \phi' = 0 \quad (5) \]
The general solution of equation (5) can be written as
\[ \phi(\eta) = a_1 + a_2 \Gamma(1/3, \frac{2\eta^3}{3}) \quad (6) \]
where \( a_1 \) and \( a_2 \) are constants and \( \Gamma \) is the incomplete Gamma function defined by
\[ \Gamma(1/3, \frac{2s^3}{3x}) = \int_{\frac{2s^3}{3x}}^{\infty} t^{-2/3} \exp(-t) dt \]
The solution of (4) with initial condition \( \delta(0) = 0 \) is
\[ \delta(x) = (2x)^{1/3}. \]
Then
\[ \eta = \frac{s}{\delta} = \frac{s}{(2x)^{1/3}} \]
From (6) we obtain
\[ \phi(x,s) = a_1 + a_2 \Gamma(1/3, \frac{s^3}{3x}) \]
Therefore,
\[ T_{0}^{in} = T_w - a_2 \Gamma(1/3, \frac{s^3}{3x}) \quad (7) \]
Therefore the boundary conditions are
\[ T_{0}^{in}(x, s) = T_w \quad \text{for } x > 0 \text{ and } s = 0 \]
\[ T_{0}^{in}(x, s) = T_w - a_2 \Gamma(1/3) \quad \text{for } x > 0 \text{ and } s \to \infty \]
\[ T_{0}^{in}(s, s) = T_w - a_2 \Gamma(1/3) \quad \text{for } s > 0 \text{ and } x \to 0 \]
And the Prandtl's matching condition is
\[ \lim_{s \to \infty} T_{0}^{in}(x, s) = \lim_{y \to 0} T_{0}^{out}(x, y) \]
Now $T_{in}^0$ is

$$
T_{in}^0(x, y) = T_0 + \frac{T_w - T_0}{\Gamma(1/3)} \Gamma(1/3, \frac{y^3}{3\varepsilon x})
$$

(8)

And thus we can write the composite one-term approximation as

$$
T^{comp} = T_0 + (T_w - T_0)e^{-(1-x)y/\varepsilon} + \frac{(T_w - T_0)}{\Gamma(1/3)} \Gamma(1/3, \frac{y^3}{3\varepsilon x})
$$

(9)

$$
+ \frac{(T_w - T_0)}{\Gamma(1/3)} \Gamma(1/3, \frac{(h - y)^3}{3\varepsilon x}) + \ldots
$$

Example 3.2. We have $T' = u, \varepsilon u' = yu$. Consider the system

$$
\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2)
$$

(10)

where $f_1$ and $f_2$ are $C^1$ functions in a simply connected domain $D \subset \mathbb{R}^2$. If $x_1 = T, x_2 = u, \frac{\partial \varphi}{\partial x_1} = 0, c(x_1, x_2) = y/\varepsilon + yu/\varepsilon > 0$ with $u > -1$

$$
f_1 \frac{\partial \varphi}{\partial x_1} + f_2 \frac{\partial \varphi}{\partial x_2} = \varphi \left(c(x_1, x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right)
$$

(11)

Hence $\varphi = e^u$ and the system does not have periodic orbits. And $\frac{dT}{du} = \frac{\varepsilon}{y}$. From $\varepsilon u' = yu$ then $u = Be^{yx/\varepsilon}$ and $T' = u$, we have $T = A + \varepsilon Be^{yx/\varepsilon}/y$

Acknowledgements. The authors express their deep gratitude to Universidad de Cartagena for partial financial support.

References


Received: October 28, 2014, Published: November 20, 2014