An Approximation to the Boussinesq Equations

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Abstract

We study the system of type Boussinesq which models the movement of a wave in the surface of a fluid in a channel. We found solutions by using Fourier transform and Neumann series.

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1 Introduction

In [2], traveling-wave solutions of Boussinesq systems was studied. It was constructed approximate and exact solutions for Boussinesq equations using homotopy perturbation Pade technique in [5]. Methods and applications to water waves were studied in [6] and [3]. In [1] was treated Boussinesq Equations. In paper [4] was obtained the solitary wave solutions of two different forms of Boussinesq equations. In this paper we study new approximated solutions of the Boussinesq equations for a small parameter $\alpha > 0$

$$\eta_t + ((1 + \alpha \eta)u)_\xi = 0,$$  
$$u_t + \eta_\xi + \frac{\alpha}{2}(u^2)_\xi - \frac{\beta}{3}u_{\xi \xi t} = 0,$$ 

subject to the condition

$$u(\xi, 0) = f(\xi), \eta(\xi, 0) = \eta_0(\xi).$$

2 Exact solution in the nonlinear case

Denoting the Fourier transform of a function $g(\xi, t)$ with respect to $\xi$ by

$$\hat{g}(k, t) = \int_{-\infty}^{\infty} g(\xi, t)e^{-ik \xi} d\xi$$

Taking the Fourier transform in (1)-(3) we obtain

$$\hat{\eta}_t + ik\hat{u} + \alpha ik \hat{\eta} \ast \hat{u} = 0, \xi \in \mathbb{R}, t \geq 0,$$  
$$(1 + \frac{\beta}{3}k^2)\hat{u}_t + ik\hat{\eta} + \alpha \hat{u} \ast iq\hat{u} = 0,$$ 

subject to the condition

$$\hat{u}(k, 0) = \hat{f}(k), \hat{\eta}(k, 0) = \hat{\eta}_0(k).$$

Taking the derivative with respect to $t$ in (5) we obtain

$$(1 + \frac{\beta}{3}k^2)\hat{u}_{tt} + ik\hat{\eta}_t + \alpha(\hat{u} \ast iq\hat{u})_t = 0,$$ 

Multiplying by $-ik$ to equation (4) and adding to equation (7) we obtain

$$(1 + \frac{\beta}{3}k^2)\hat{u}_{tt} + k^2\hat{u} + \alpha k^2 \hat{\eta} \ast \hat{u} + \alpha(\hat{u} \ast iq\hat{u})_t = 0$$
Dividing by \( k^2 \) we have

\[
\frac{1 + \beta k^2}{k^2} \hat{u}_{tt} + \hat{u} + \alpha \hat{\eta} \ast \hat{u} + \frac{\alpha}{k^2} (\hat{u} \ast iq \hat{u})_t = 0 \tag{9}
\]

the solution is

\[
\hat{u}(k,t) = c_1 e^{-ik \sqrt{1 + \frac{\beta}{3} k^2} t} + c_2 e^{ik \sqrt{1 + \frac{\beta}{3} k^2} t} + \alpha T \hat{u} \tag{10}
\]

where

\[
f(t) = \hat{\eta} \ast \hat{u} + \frac{1}{k^2} (\hat{u} \ast iq \hat{u})_t \tag{11}
\]

Dividing by \( ik \) in (5) we obtain

\[
\hat{\eta} = -\frac{1}{ik} ((1 + \frac{\beta}{3} k^2) \hat{u}_t + \alpha \hat{u} \ast i q \hat{u}) \tag{12}
\]

Convoluting with \( \hat{u} \)

\[
\hat{\eta} \ast \hat{u} = -\frac{1}{ik} ((1 + \frac{\beta}{3} k^2) \hat{u}_t + \alpha \hat{u} \ast i q \hat{u}) \ast \hat{u} \tag{13}
\]

Then \( f(t) \) depends on \( \hat{u} \) and from (10)

\[
\hat{u}(k,t) = c_1 e^{-ik \sqrt{1 + \frac{\beta}{3} k^2} t} + c_2 e^{ik \sqrt{1 + \frac{\beta}{3} k^2} t} + \alpha T \hat{u}
\]

where

\[
T \hat{u} = e^{ik \sqrt{1 + \frac{\beta}{3} k^2} t} \int_0^t \frac{i \sqrt{1 + \frac{\beta}{3} k^2} f(\tau) e^{-ik \sqrt{1 + \frac{\beta}{3} k^2} \tau}}{2k(1 + \frac{\beta}{3} k^2)} d\tau
\]

\[
+ e^{-ik \sqrt{1 + \frac{\beta}{3} k^2} t} \int_0^t \frac{i \sqrt{1 + \frac{\beta}{3} k^2} f(\tau) e^{ik \sqrt{1 + \frac{\beta}{3} k^2} \tau}}{2k(1 + \frac{\beta}{3} k^2)} d\tau
\]
For shallow water waves of small amplitude we have that $\alpha$ and $\beta$ are very small where $\alpha$ is related with nonlinear term and $\beta$ with respect to the dispersive term and they are dimensionless parameters. Taking $\alpha \to 0$ and using the fact that $(1 - \alpha T)^{-1}$ there exists, then applying Neumann series we have

$$(1 - \alpha T)^{-1} = \sum_{n=0}^{\infty} \alpha^n T^n.$$ 

Then from (13)

$$\hat{u}(k, t) = \sum_{n=0}^{\infty} \alpha^n T^n \hat{u}(c_1 e^{-\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}} + c_2 e^{\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}})$$

Using the inverse Fourier theorem we obtain

$$u(\xi, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \alpha^n T^n \hat{u}(c_1 e^{-\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}} + c_2 e^{\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}}) e^{ik\xi} d\xi$$

This is an approximation of $u$ as $\alpha \to 0$.

If we take the derivative with respect to $t$ in this equation we obtain

$$\hat{u}_t(k, t) = c_1 \left( -\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}} \right) e^{-\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}} + c_2 \left( \frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}} \right) e^{\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}} + (\sum_{n=1}^{\infty} \alpha^n T^n \hat{u}(c_1 e^{-\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}} + c_2 e^{\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}}))_t$$

And then replacing this into (12) we get

$$\hat{\eta} = c_1 \sqrt{1 + \frac{\beta}{3}k^2} e^{-\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}} - c_2 \sqrt{1 + \frac{\beta}{3}k^2} e^{\frac{ik}{\sqrt{1 + \frac{\beta}{3}k^2}}} + O(\alpha)$$

Now using the initial conditions (6) we obtain

$$c_1 = \frac{\tilde{f}(k)}{2} + \frac{\hat{\eta}_0(k)}{2 \sqrt{1 + \frac{\beta}{3}k^2}} + O(\alpha)$$

$$c_2 = \frac{\tilde{f}(k)}{2} - \frac{\hat{\eta}_0(k)}{2 \sqrt{1 + \frac{\beta}{3}k^2}} + O(\alpha)$$

We have proved the following theorem

**Theorem 2.1.** The solution of the problem (1) - (3) is given by (14), (16) and (17).

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References


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