International Journal of Mathematical Analysis
Vol. 8, 2014, no. 49, 2445 - 2450
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2014.49275

A Generalized Nadler's Theorem in Dislocated Quasi-Metric spaces

Priscilla S. Macansantos

University of the Philippines Baguio
Baguio City 2600, Philippines

Abstract

Fixed Point theorems in partial metric spaces, weak partial metric spaces and metric-like spaces have been the subject of recent work, with the interest generated in partial metric spaces and its extension/s (as a suitable structure for studies in theoretical computer science). Several approaches to fixed point theory for point-valued functions on complete metric spaces have been generalized to partial metric spaces, weak partial metric spaces, and dislocated quasi-metric spaces (see, for instance, Alghamdi [1], Ahmed [2]). For set-valued functions, substantial work may still be done to generalize the theory in the setting of partial metric spaces, and other weaker metric-like spaces, which are encountered in applications to areas such as theoretical computer science. Recently, M.A. Ahmed [2] established a theorem on fixed points for continuous dq-closed valued multifunctions which are generalized k-contractions on a complete subspace of a dq-metric space. In this paper we take off from Ahmed and proceed to establish a similar result but with a more general contraction condition as well as a weaker condition of upper-semicontinuity of the multivalued mapping. As a consequence of our generalization, we are able to include as special cases the theorem of Ahmed.

Mathematics Subject Classification: 54H25

Keywords: Dislocated quasi-metric space, Multivalued mappings; Fixed Point theorem

1 Introduction

Banach's fixed point theorem for contraction mappings on complete metric spaces is a key result used in theoretical and applied mathematics. For instance, establishing the existence of solutions to differential and integral equations is made possible through an application of the theorem to the appropriate function space. Subsequent work in Fixed Point Theory has given rise to many generalizations of Banach's theorem. Of note is the generalization of Nadler [14] to multifunctions on metric spaces satisfying a contraction condition. Some recent treatments and extensions of fixed point theorems for multivalued functions are given in [2], [3], and [4] .

The concept of partial metric spaces as a generalization of metric spaces was introduced in 1994 by Mathews [12], in his treatment of denotational semantics of dataflow networks. For partial metric spaces, self-distance need not be 0. Some applications of partial metrics to problems in theoretical computer science, including the use of fixed point theorems to determine program output from partially defined information, are cited in X. Huang et al [8] and references therein. Heckmann [13] further generalized this notion to that of weak partial metric spaces. As it turns out, yet other generalizations are possible, on considering other distance spaces, such as dislocated quasi-metric spaces.

In [4] we used the approach of Damjanovic et al [5] who looked into pairs of multi-valued and single-valued maps in complete metric spaces, and used coincidence and common fixed points, to establish a theorem on fixed points for pairs of multivalued functions. In that paper, we proved a similar result, in the setting of partial metric spaces. In a recent paper by Ahmed [2], dislocated quasi-metric spaces (a class of spaces which include partial metric spaces, satisying even weaker conditions) are defined, and a fixed point theorem for continuous generalized k-contractions on dislocated quasi-metric spaces is established. In this paper we proceed to establish a similar result but with a more general contraction condition as well as a weaker condition of upper-semicontinuity on the multivalued mapping. As a consequence of our generalization, we are able to include as special cases the theorem of Ahmed.

The paper is organized as follows. In Section 2, some basic definitions which will be used later in the paper are provided. In Section 3, we present the main theorem and some corollaries.

2 Preliminaries

Definition 2.1 *Let X be a nonempty set. A function $d : X \times X \to \Re^+$ is said to be a metric on X if for any $x, y, z \in X$, the following conditions hold:*

(d1) $d(x, x) = 0$ *;*

(d2) $d(x, y) = d(y, x) = 0$ *implies* $x = y$*;*

(d3) $d(x, y) = d(y, x)$*;*

(d4) $d(x, z) \leq d(x, y) + d(y, z)$*.*

The pair (X, d) *is then called a metric space. If* d *satisfies* $(d2), (d3)$ *and* $(d4)$*,* (X, d) *is a dislocated metric space (d-metric or metric-like space).*
If d *satisfies* $(d1), (d2)$ *and* $(d4)$*,* (X, d) *is a quasi-metric space (q-metric)*
If only $(d2)$ *and* $(d4)$ *are met, then* (X, d) *is then called a dislocated quasi-metric space (dq-metric)*

Definition 2.2 *Let* X *be a nonempty set. A function* $p : X \times X \to \Re^+$ *is said to be a partial metric on* X *if for any* $x, y, z \in X$*, the following conditions hold:*

(P1) $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$*;*

(P2) $p(x, x) \leq p(x, y)$*;*

(P3) $p(x, y) = p(y, x)$*;*

(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$*.*

The pair (X, p) *is then called a partial metric space.*

If $p(x, y) = 0$, then $x = y$. But the converse does not always hold. A standard example of a partial metric space is the pair $(\Re^+, p)$, where $p : \Re^+ \times \Re^+ \to \Re^+$ is defined as $p(x, y) = max\{x, y\}$. It is noted that any partial metric space is d-metric, and any d-metric is dq-metric.

In the sequel, X will refer to a dq-metric space with distance function d. Following Ahmed, for $B \subset X$, define the following

Definition 2.3 $d(A, B) = inf\{d(a, b) : a \in A, b \in B\}$
$\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}$
$H(A, B) = max\{sup_{a \in A} d(a, B), sup_{b \in B} d(A, b)\}$
If $A = \{a\}$*, then* $d(a, B) = inf\{d(a, b) : b \in B\}$*.*

Definition 2.4 *The dq-closure of* A *is given by (dq-cl)*$(A) = \{x \in X : A \neq \emptyset, d(x, A) = 0\}$*. Further,* A *is dq-closed iff (dq-cl)*$(A) \subseteq A$*.*

Definition 2.5 *A sequence* $\{x_n\}$ *in a dq-metric space* (X, d) *converges to a point* $x \in X$ *if and only if* $lim_{n \to \infty} d(x_n, x) = 0$ *or* $lim_{n \to \infty} d(x, x_n) = 0$*.*

Definition 2.6 *Let* (X, d) *be a dq-metric space. (a) A sequence* $\{x_n\}$ *in* X *is said to be a Cauchy sequence if* $lim_{n,m \to \infty} d(x_n, x_m)$ *exists and is finite. (b)* (X, d) *is said to be complete if every Cauchy sequence* x_n *in* X *converges to a point* $x \in X$

From Ahmed, we have the following theorems:

Theorem 2.7 *Let (X, d) be a dq-metric space. Then every convergent sequence in X is Cauchy.*

Theorem 2.8 *Let (X, d) be a dq-metric space. If $\{x_n\}$ is a sequence in X converging to $x \in X$, then every subsequence of $\{x_n\}$ converges to x.*

The proofs are similar to that given for partial metric spaces.

Finally, we define upper-semicontinuity for multivalued mappings:

Definition 2.9 *A multivalued function $F : X \to 2^X$ is upper-semicontinuous if for every sequence $\{x_n\}$ converging to x in X, and for all $y_n \in F(x_n)$ for which $y_n \to y$, then $y \in F(x)$.*

3 Main Results

The version of the following lemma for partial metric spaces was proven in Aydi [3]. The proof for this lemma is the same as that in Aydi.

Lemma 3.1 *Let (X, d) be a dislocated quasi-metric space, $A, B \in CB(X)$ and $\lambda > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that $d(a, b) \leq \lambda H(A, B)$.*

Using this, we now give our main result:

Theorem 3.2 *Let (X, d) be a complete dq-metric space and let $T : X \to CB(X)$ be a multi-valued map that is upper-semicontinuous. Suppose that for all $x, y \in X$,*

$H(Tx, Ty) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)]$, $\alpha, \beta \geq 0$ *and* $0 < \alpha + 2\beta < 1$.
Then, there exists $u \in X$ such that $u \in T(u)$.

In the above theorem, $CB(X)$ refers to closed and bounded subsets of X. Boundedness in dq-metric spaces are the same as in ordinary metric spaces, i.e, $A \subseteq X$ is bounded if and only if there is some $v \in A$ and a finite M such that for all $a \in A$, $d(v, a) \leq M$.

Proof

Since $0 < \alpha + 2\beta < 1$, there exists $r > 0$ such that $\alpha + 2\beta < \sqrt{r} < 1$. Note that $\lambda := 1/\sqrt{r} > 1$. Let $\theta := \frac{\alpha+\beta}{\sqrt{r}-\beta}$. clearly, $0 < \theta < 1$. Let $x_0 \in X$. There exists $x_1 \in X$ with $x_1 \in T(x_0)$. From the Lemma, we can find $x_2 \in T(x_1)$ with $d(x_1, x_2) \leq \lambda H(T(x_0), T(x_1))$. But $H(T(x_0), T(x_1)) \leq \alpha d(x_0, x_1) + \beta[d(x_0, T(x_0)) + d(x_1, T(x_1))]$, so $d(x_1, x_2) \leq \alpha d(x_0, x_1) + \beta[d(x_0, T(x_0)) + d(x_1, T(x_1))]$. Now $d(x_0, T(x_0)) \leq d(x_0, x_1)$ and $d(x_1, T(x_1)) \leq d(x_1, x_2)$, hence, $d(x_1, x_2) \leq \lambda\{\alpha d(x_0, x_1) + \beta[d(x_0, x_1)) + d(x_1, x_2)]\}$. This yields $(1 - \lambda\beta)d(x_1, x_2) \leq \lambda(\alpha + \beta)d(x_0, x_1)$.

Hence, $d(x_1, x_2) \leq \theta d(x_0, x_1)$. Using the same argument, we find $x_3 \in T(x_2)$ such that $d(x_2, x_3) \leq \theta d(x_1, x_2) \leq \theta^2 d(x_0, x_1)$.

Applying the argument repeatedly, we get $x_{n+1} \in T(x_n)$ for which
$d(x_n, x_{n+1}) \leq \theta d(x_{n-1}, x_n) \leq \theta^n d(x_0, x_1)$. Hence,
$d(x_n, x_m) = d(x_n, x_{n+q}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+q-1}, x_{n+q})$
$\leq \{\theta^n + \theta^{n+1} + \ldots + \theta^{n+q-1}\}d(x_0, x_1) \leq \frac{\theta^n}{1-\theta}d(x_0, x_1)$
Also, $d(x_m, x_n) = d(x_{n+q}, x_n) \leq \frac{\theta^n}{1-\theta}d(x_0, x_1)$.

Hence, $\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} d(x_m, x_n) = 0$. Consequently, $\{x_n\}$ is a Cauchy sequence, and by completeness of X, a limit $u \in X$ exists. The subsequence $\{x_{n+1}\}$ is also Cauchy and is convergent to the same limit u. Invoking the upper-semicontinuity of T, we get $u \in T(u)$.

Corollary 3.3 *The theorem here established has, as a special case, the result of Ahmed.*

Corollary 3.4 *A result, applied to continuous single-valued functions on a dq-metric space satisfying the same contractive condition also holds. A further special case of this would result in Banach's fixed point theorem on dq-metric spaces.*

ACKNOWLEDGEMENTS. The author acknowledges the support of the the University of the Philippines Baguio for research load credit during the second semester, 2013-2014.

References

[1] M. A. Alghamdi, N. Shahzad and O. Valero, On fixed point theory in partial metric spaces, *Fixed Point Theory and Applications,* 175 (2012).

[2] M. A. Ahmed, SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN DISLOCATED QUASI-METRIC SPACES, *DEMONSTRATIO MATHEMATICA, Vol. XLV* 1 (2012).

[3] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on complete partial metric spaces, *Topology and its Applications,* **159** (2012), 3234-3242.

[4] P.S. Macansantos, A Fixed Point Theorem for Multifunctions in Partial Metric spaces, *Journal of Nonlinear Analysis and Application,* (2013) 1-7.

[5] B. Damjanovic, B. Samet and C. Vetro, Common Fixed Point Theorems for Multi-Valued Maps, *Acta Mathematica Scientia,* **32B(2)** (2012) 818-824.

[6] M.E. Gordji, H. Baghani, H. Khodaei and M. Ramezani, A Generalization of Nadler's Fixed Point Theorem, *J. Nonlinear Sci. Appl.,***3** (2010), 148-151.

[7] G. Hardy and T. Rogers, A Generalization of a fixed point theorem of Reich, *Canad. Math. Bull.,* **16 (2)** (1973) 201-206.

[8] X. Huang, C. Zhu and X. Wen, Fixed point theorems for expanding mappings in partial metric spaces, *An. St. Univ. Ovidius Constanta,***20(1)**, (2012), 213-224.

[9] T. Kamran, Coincidence and fixed points for hybrid strict contractions, *J. Math. Anal. Appl.,* **299** (2004) 235-241.

[10] T. Lazar, G. Mot, G. Petrusel and S. Szentesi, The Theory of Reichs Fixed Point Theorem for Multivalued Operators, *Fixed Point Theory and Applications,* (2010) available online Article ID 178421, 10 pages doi:10.1155/2010/178421

[11] P. Macansantos, A Generalized Nadler-type theorem in partial metric spaces, *Int. Journal of Math. Analysis,* Vol. 7 (2013) No.7, 343-348.

[12] S.G. Matthews, Partial Metric Topology, in Papers on General Topology and Applications, Eighth Summer Conference at Queens College, *Annals of the New York Academy of Sciences,***728**, (1994), 183-197.

[13] R. Heckmann, Approximation of Metric Spaces by Partial Metric Spaces, *Applied Categorical Structures,***7**, (1999), 71-83.

[14] S.B. Nadler Jr., Multi-valued Contraction mappings, *Pacific J. Math.,* **30** (1969), 475 - 488.

[15] H.K. Nashine and H. Aydi, Common fixed points for generalized contractive mappings in cone metric spaces, *Journal of Nonlinear Analysis and Application,* Volume 2012 (2012) , 1-12.

[16] S. Reich, Fixed points of contractive functions, *Bollettino della Unione Matematica Italiana,* **5** (1972) 26-42.

Received: September 5, 2014; Published October 28, 2014