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Superstability of the Difference-form Equations with a Cocycle Related to Distance Measures

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Abstract

This work extends the study on the stability of the functional equation $f(pr, qs) - f(ps, qr) = f(p, q)\, f(r, s)$, which arise in the characterization of the nonsymmetrically compositive difference-form related to distance measures, products of some multiplicative functions.

In this paper, we obtain the superstability of the functional equation

$$f(pr, qs) - f(ps, qr) = \theta(pq, rs) f(p, q)\, f(r, s)$$

for all $p, q, r, s \in G$, where G is an Abelian group, f a nonzero functional on G^2, and θ a cocycle on G^2. Also we investigate the superstability with following functional equations:

$$f(pr, qs) - f(ps, qr) = \theta(pq, rs) f(p, q) g(r, s),$$
$$f(pr, qs) - f(ps, qr) = \theta(pq, rs) g(p, q) f(r, s).$$

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1. Introduction

Let $(G, \cdot)$ be an Abelian group. Let I denote the open unit interval (0, 1). Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$ be a set of positive real numbers and $\mathbb{R}_k = \{x \in \mathbb{R} \mid x > k > 0\}$ for some $k \in \mathbb{R}$.

In [1], Chung, Kannappan, Ng and Sahoo characterized all symmetrically compositive sum form distance measures with a measurable generating function. The following functional equation

$$(FE) \qquad f(pr, qs) + f(ps, qr) = f(p, q)\, f(r, s)$$

holding for all $p, q, r, s \in I$ was instrumental in their characterization. Among other results, they proved the following result giving the general solution of the functional equation (FE). *Suppose $f : I^2 \to \mathbb{R}$ satisfies (FE) for all $p, q, r, s \in I$. Then $f(p, q) = M_1(p)\, M_2(q) + M_1(q)\, M_2(p)$, where $M_1, M_2 : \mathbb{R} \to \mathbb{C}$ are multiplicative functions. Further, either M_1 and M_2 are both real or M_2 is the complex conjugate of M_1. The converse is also true.*

The stability of the functional equation (FE) and four generalizations of (FE) namely, $f(pr, qs) + f(ps, qr) = f(p, q)\, g(r, s)$, $f(pr, qs) + f(ps, qr) = g(p, q)\, f(r, s)$, $f(pr, qs) + f(ps, qr) = g(p, q)\, g(r, s)$, $f(pr, qs) + f(ps, qr) = g(p, q)\, h(r, s)$ for all $p, q, r, s \in G$, were studied in [6] and [7].

The following functional equation

$$(DM) \qquad f(pr, qs) - f(ps, qr) = f(p, q)\, f(r, s)$$

holding for all $p, q, r, s \in I$ arises in the characterization of the nonsymmetrically compositive difference-form related to distance measures, products of some multiplicative functions. The stability of (DM) is investigated in Kim [5]

The present work continues the study for the stability of the Pexider type functional equations of (FE) and (DM) added a cocycle property to the conditions in the results ([6], [7], [5], [8]).

J. Tabor [13] investigated the cocycle property. The definition of cocycle as follows;

Definition 1. *A function $\theta : G^2 \to \mathbb{R}$ is a cocycle if it satisfies the equation*

$$\theta(a, bc)\theta(b, c) = \theta(ab, c)\theta(a, b), \quad \forall\, a, b, c \in G.$$

For example, if $F(x, y) = \frac{f(x)f(y)}{f(xy)}$ for a function $f : \mathbb{R} \to \mathbb{R}_+$, then F is a cocycle. Also if $\theta(x, y) = \ln(x)\ln(y)$ for a function $\theta : {\mathbb{R}_+}^2 \to (\mathbb{R}, +)$, then θ is a cocycle, that is, $\theta(a, bc) + \theta(b, c) = \theta(ab, c) + \theta(a, b)$, and in this case, it is well known that $\theta(x, y)$ is represented by $B(x, y) + M(xy) - M(x) - M(y)$ where B is an arbitrary skew-symmetric biadditive function and M is some function [2]. If $\theta(x, y) = a^{\ln(x)\ln(y)}$, then $\theta : {\mathbb{R}_+}^2 \to (\mathbb{R}, \cdot)$ is a cocycle and in this case, $\theta(x, y)$ is represented by $e^{B(x,y)} e^{M(xy)-M(x)-M(y)}$.

Let us consider the functional equations of a symmetrically compositive sum-form and a nonsymmetrically compositive difference-form related to distance measures with a cocycle:

$$(CSM) \qquad f(pr, qs) + f(ps, qr) = \theta(pq, rs)\, f(p, q)\, f(r, s)$$

$$(CDM) \qquad f(pr, qs) - f(ps, qr) = \theta(pq, rs)\, f(p, q)\, f(r, s)$$

for all $p, q, r, s \in G$ and where f, θ are nonzero functionals on G^2, which can be represented as exponential functional equation in reduction.

In fact, if $f(x, y) = \frac{1}{x} + \frac{1}{y}$, then $f(pr, qs) + f(ps, qr) = f(p, q)\, f(r, s)$, and also if $f(x, y) = a^{\ln xy}$, and $\theta(x, y) = 2$ then f, θ satisfy the equation $f(pr, qs) + f(ps, qr) = \theta(pq, rs)\, f(p, q)\, f(r, s)$.

This paper aims to investigate the superstability of (CDM) and two generalized functional equations of (CDM), namely, as well as that of the following type functional equations:

$$(CDM_{fffg}) \qquad f(pr, qs) - f(ps, qr) = \theta(pq, rs) f(p, q) g(r, s),$$

$$(CDM_{ffgf}) \qquad f(pr, qs) - f(ps, qr) = \theta(pq, rs) g(p, q) f(r, s).$$

2. Superstability of the equations

In this section, the theorems state that an approximate equations of each (CDM), (CDM_{fffg}), and (CDM_{ffgf}) with the boundedness conditions of $f(p, q) \pm g(p, q)$ also imply the functional equation (CDM), (CDM_{fffg}), and (CDM_{ffgf}), respectively.

Theorem 1. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$|f(pr, qs) - f(ps, qr) - \theta(pq, rs) g(p, q) f(r, s)| \le \phi(r, s) \quad \forall\, p, q, r, s \in G. \tag{2.1}$$

and $|f(p, q) - g(p, q)| \le M$ for all $p, q \in G$ and some constant M.

Then either g is bounded or f satisfies (CDM). In particular, if g satisfies (CDM), then f and g satisfy (CDM_{ffgf}) without above bounded condition M.

Proof. Let g be an unbounded solution of inequality (2.1). Then there exists a sequence $\{(x_n, y_n) | n \in N\}$ in G^2 such that $0 \neq |g(x_n, y_n)| \to \infty$ as $n \to \infty$.

Letting $p = x_n$, $q = y_n$ in (2.1) and dividing by $|\theta(x_n y_n, rs) g(x_n, y_n)|$, we have

$$\Big| \frac{f(x_n r, y_n s) - f(x_n s, y_n r)}{\theta(x_n y_n, rs) g(x_n, y_n)} - f(r, s) \Big| \le \frac{\phi(r, s)}{k|g(x_n, y_n)|}.$$

Passing to the limit as $n \to \infty$, we obtain

$$f(r, s) = \lim_{n\to\infty} \frac{f(x_n r, y_n s) - f(x_n s, y_n r)}{\theta(x_n y_n, rs) g(x_n, y_n)}. \tag{2.2}$$

Letting $p = x_n p$, $q = y_n q$ in (2.1) and dividing by $|g(x_n, y_n)|$, we have

$$\Big| \frac{f(x_n pr, y_n qs) - f(x_n ps, y_n qr)}{g(x_n, y_n)} - \frac{\theta(x_n p y_n q, rs) g(x_n p, y_n q)}{g(x_n, y_n)} f(r, s) \Big| \le \frac{\phi(r, s)}{|g(x_n, y_n)|} \to 0 \tag{2.3}$$

as $n \to \infty$.

Letting $p = x_n q$, $q = y_n p$ in (2.1) and dividing by $|g(x_n, y_n)|$, we have

$$\Big|\frac{f(x_nqr, y_nps) - f(x_nqs, y_npr)}{g(x_n, y_n)} - \frac{\theta(x_nqy_np, rs)g(x_nq, y_np)}{g(x_n, y_n)} f(r,s)\Big| \leq \frac{\phi(r,s)}{|g(x_n, y_n)|} \to 0 \tag{2.4}$$

as $n \to \infty$.

Note that for any a, b, c in G, $\theta(ba, c)\theta(b, a) = \theta(b, ac)\theta(a, c)$ by the definition of the cocycle. Letting $pq = a$, $x_ny_n = b$, and $rs = c$ we have

$$\frac{\theta(x_ny_npq, rs)\theta(x_ny_n, pq)}{\theta(x_ny_n, pqrs)} = \theta(pq, rs)$$

for any p, q, r, s, x_n, y_n in G.

Thus, from (2.2), (2.3), and (2.4), we obtain

$$\begin{aligned}
&\Big|f(pr, qs) - f(ps, qr) - \theta(pq, rs)f(p,q)f(r,s)\Big| \\
&= \lim_{n\to\infty}\Big|\frac{f(x_npr, y_nqs) - f(x_nqs, y_npr) - f(x_nps, y_nqr) + f(x_nqr, y_nps)}{\theta(x_ny_n, prqs)g(x_n, y_n)} \\
&\qquad - \theta(pq, rs)f(p,q)f(r,s)\Big| \\
&\leq \lim_{n\to\infty}\Big|\frac{1}{\theta(x_ny_n, prqs)}\Big| \cdot \Big|\frac{f(x_npr, y_nqs) - f(x_nps, y_nqr)}{g(x_n, y_n)} \\
&\qquad - \frac{\theta(x_npy_nq, rs)g(x_np, y_nq)f(r,s)}{g(x_n, y_n)}\Big| \\
&\quad + \lim_{n\to\infty}\Big|\frac{1}{\theta(x_ny_n, prqs)}\Big| \cdot \Big|\frac{\theta(x_nqy_np, rs)g(x_nq, y_np)f(r,s)}{g(x_n, y_n)} \\
&\qquad - \frac{f(x_nqs, y_npr) - f(x_nqr, y_nps)}{g(x_n, y_n)}\Big| \\
&\quad + |f(r,s)| \lim_{n\to\infty}\Big|\frac{\theta(x_ny_npq, rs)\theta(x_ny_n, pq)}{\theta(x_ny_n, pqrs)} \cdot \frac{g(x_np, y_nq) - g(x_nq, y_np)}{\theta(x_ny_n, pq)g(x_n, y_n)} \\
&\qquad - \theta(pq, rs)f(p,q)\Big| \qquad (2.5) \\
&= |f(r,s)|\theta(pq, rs) \lim_{n\to\infty}\Big|\frac{f(x_np, y_nq) - f(x_nq, y_np)}{\theta(x_ny_n, pq)g(x_n, y_n)} \\
&\qquad + \frac{(g-f)(x_np, y_nq) + (f-g)(x_nq, y_np)}{\theta(x_ny_n, pq)g(x_n, y_n)} - f(p,q)\Big| \\
&\leq |f(r,s)|\theta(pq, rs) \lim_{n\to\infty}\Big|\frac{2M}{kg(x_n, y_n)}\Big| \\
&\quad + |f(r,s)|\theta(pq, rs) \lim_{n\to\infty}\Big|\frac{f(x_np, y_nq) - f(x_nq, y_np)}{\theta(x_ny_n, pq)g(x_n, y_n)} - f(p,q)\Big| \\
&= 0.
\end{aligned}$$

In particular, if g satisfies (CSM), then each term of inequality (2.5) converges to zero before arriving at bounded condition M. Indeed, we have

$$
\begin{aligned}
&\Big|f(pr,qs)-f(ps,qr)-\theta(pq,rs)g(p,q)f(r,s)\Big|\\
&=\lim_{n\to\infty}\Big|\frac{f(x_npr,y_nqs)-f(x_nqs,y_npr)-f(x_nps,y_nqr)+f(x_nqr,y_nps)}{\theta(x_ny_n,prqs)g(x_n,y_n)}\\
&\qquad -\theta(pq,rs)g(p,q)f(r,s)\Big|\\
&\le\lim_{n\to\infty}\Big|\frac{1}{\theta(x_ny_n,prqs)}\Big|\cdot\Big|\frac{f(x_npr,y_nqs)-f(x_nps,y_nqr)}{g(x_n,y_n)}\\
&\qquad -\frac{\theta(x_npy_nq,rs)g(x_np,y_nq)f(r,s)}{g(x_n,y_n)}\Big|\\
&\quad+\lim_{n\to\infty}\Big|\frac{1}{\theta(x_ny_n,prqs)}\Big|\cdot\Big|\frac{\theta(x_nqy_np,rs)g(x_nq,y_np)f(r,s)}{g(x_n,y_n)}\\
&\qquad -\frac{f(x_nqs,y_npr)-f(x_nqr,y_nps)}{g(x_n,y_n)}\Big|\\
&\quad+|f(r,s)|\lim_{n\to\infty}\Big|\frac{\theta(x_ny_npq,rs)\theta(x_ny_n,pq)}{\theta(x_ny_n,pqrs)}\cdot\frac{g(x_np,y_nq)-g(x_nq,y_np)}{\theta(x_ny_n,pq)g(x_n,y_n)}\\
&\qquad -\theta(pq,rs)g(p,q)\Big|=0,
\end{aligned}
$$

It obtains that f and g satisfy (CDM_{ffgf}). □

Theorem 2. *Let $f,g:G^2\to\mathbb{R}$, $\phi:G^2\to\mathbb{R}_+$ be functions and a function $\theta:G^2\to\mathbb{R}_k$ be a cocycle satisfying*

$$
|f(pr,qs)-f(ps,qr)-\theta(pq,rs)g(p,q)f(r,s)|\le\phi(p,q)\quad\forall\ p,q,r,s\in G. \tag{2.6}
$$

Then either f is bounded or g satisfies (CSM).

Proof. For f to be an unbounded solution of inequality (2.6), we can choose a sequence $\{(x_n,y_n)|n\in N\}$ in G^2 such that $0\ne|f(x_n,y_n)|\to\infty$ as $n\to\infty$.

Letting $r=x_n$, $s=y_n$ in (2.6) and dividing by $|\theta(pq,x_ny_n)f(x_n,y_n)|$, we have

$$
\Big|\frac{f(px_n,qy_n)-f(py_n,qx_n)}{\theta(pq,x_ny_n)f(x_n,y_n)}-g(p,q)\Big|\le\frac{\phi(p,q)}{k|f(x_n,y_n)|}.
$$

Passing to the limit as $n\to\infty$, we obtain that

$$
g(p,q)=\lim_{n\to\infty}\frac{f(px_n,qy_n)-f(py_n,qx_n)}{\theta(pq,x_ny_n)f(x_n,y_n)}. \tag{2.7}
$$

Replacing $r=rx_n$, $s=sy_n$ in (2.6) and dividing by $|f(x_n,y_n)|$, we have

$$(2.8)\qquad \Big|\frac{f(prx_n, qsy_n) - f(psy_n, qrx_n)}{f(x_n, y_n)} - \theta(pq, rx_n sy_n)g(p,q)\frac{f(rx_n, sy_n)}{f(x_n, y_n)}\Big| \leq \frac{\phi(p,q)}{|f(x_n, y_n)|} \to 0$$

as $n \to \infty$.

Replacing $r = ry_n$, $s = sx_n$ in (2.6) and dividing by $|f(x_n, y_n)|$, we have

$$(2.9)\qquad \Big|\frac{f(pry_n, qsx_n) - f(psx_n, qry_n)}{f(x_n, y_n)} - g(p,q)\theta(pq, ry_n sx_n)\frac{f(ry_n, sx_n)}{f(x_n, y_n)}\Big| \leq \frac{\phi(p,q)}{|f(x_n, y_n)|} \to 0$$

as $n \to \infty$.

Thus from (2.7), (2.8), and (2.14), we obtain

$$\begin{aligned}
&\Big|g(pr, qs) + g(ps, qr) - \theta(pq, rs)g(p,q)g(r,s)\Big| \\
&= \lim_{n\to\infty}\Big|\frac{f(prx_n, qsy_n) - f(pry_n, qsx_n) + f(psx_n, qry_n) - f(psy_n, qrx_n)}{\theta(prqs, x_n y_n)f(x_n, y_n)} \\
&\qquad - \theta(pq, rs)g(p,q)g(r,s)\Big| \\
&\leq \lim_{n\to\infty}\Big|\frac{1}{\theta(pqrs, x_n y_n)}\Big| \cdot \Big|\frac{f(prx_n, qsy_n) - f(psy_n, qrx_n)}{f(x_n, y_n)} \\
&\qquad - g(p,q)\theta(pq, rx_n sy_n)\frac{f(rx_n, sy_n)}{f(x_n, y_n)}\Big| \\
&\quad + \lim_{n\to\infty}\Big|\frac{1}{\theta(pqrs, x_n y_n)}\Big| \cdot \Big|g(p,q)\theta(pq, ry_n sx_n)\frac{f(ry_n, sx_n)}{f(x_n, y_n)} \\
&\qquad - \frac{f(pry_n, qsx_n) - f(psx_n, qry_n)}{f(x_n, y_n)}\Big| \\
&\quad + |g(p,q)|\lim_{n\to\infty}\Big|\frac{\theta(pq, rx_n sy_n)\theta(rs, x_n y_n)}{\theta(pqrs, x_n y_n)} \cdot \frac{f(rx_n, sy_n) - f(ry_n, sx_n)}{\theta(rs, x_n y_n)f(x_n y_n)} \\
&\qquad - \theta(pq, rs)g(r,s)\Big| = 0.
\end{aligned}$$

The proof of the theorem is now complete. □

The proofs of Theorem 3 and Theorem 4 run to go through the same procedure as step by step in Theorem 1 and Theorem 2.

Theorem 3. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$(2.10)\quad |f(pr, qs)-f(ps, qr)-\theta(pq, rs)f(p, q)g(r, s)| \leq \phi(r, s) \quad \forall\, p, q, r, s \in G.$$

Then either f is bounded or g satisfies (CDM).

Theorem 4. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$(2.11)\quad |f(pr, qs)-f(ps, qr)-\theta(pq, rs)f(p, q)g(r, s)| \leq \phi(p, q) \quad \forall\, p, q, r, s \in G,$$

and $|f(p, q) - g(p, q)| \leq M$ for all $p, q \in G$ and some constant M.

Then, either g is bounded or f satisfies (CSM). In particular, if g satisfies (CDM), then f and g satisfy (CDM_{fffg}) without above bounded condition M.

The proof of the below Theorem 5 follows immediately from Theorem 1 and Theorem 2 (or Theorem 3 and Theorem 4) .

Theorem 5. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$(2.12)\qquad |f(pr, qs) - f(ps, qr) - \theta(pq, rs)f(p, q)f(r, s)| \leq \begin{cases} (i) & \phi(r, s) \\ (ii) & \phi(p, q) \end{cases}$$

(a) In case (i), then, either f is bounded or f satisfies (CDM).
(b) In case (ii), then, either f is bounded or f satisfies (CSM).

Replacing $\phi(r, s)$ and $\phi(p, q)$ by ε in the obtained results, then it imply the following corollaries.

Corollary 1. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$(2.13)\qquad |f(pr, qs) - f(ps, qr) - \theta(pq, rs)g(p, q)f(r, s)| \leq \varepsilon \quad \forall\, p, q, r, s \in G.$$

and $|f(p, q) - g(p, q)| \leq M$ for all $p, q \in G$ and some constant M.

Then, either g is bounded or f satisfies (CDM). In particular, if g satisfies (CDM), then f and g satisfy (CDM_{ffgf}) without above bounded condition M.

Corollary 2. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$(2.14)\qquad |f(pr, qs) - f(ps, qr) - \theta(pq, rs)g(p, q)f(r, s)| \leq \varepsilon \quad \forall\, p, q, r, s \in G.$$

Then f (or g) is bounded or g satisfies (CSM). And also f and g satisfy (CDM_{ffgf}).

Proof. From Theorem 2, we find that f is bounded or g satisfies (CSM). For the case g is bounded, it is sufficent to show that f is bounded implies g is bounded. Indeed, we can see it from (2.14), for all $p, q, r, s \in G$

$$|g(p, q)| \leq \frac{\varepsilon + |f(pr, qs) - f(ps, qr)|}{k|f(r, s)|}.$$

If g be unbounded, then f is unbounded. Hence g also satisfies (CSM). From the second part of Corollary 1, f and g satisfy (CDM_{ffgf}). $\square$

Corollary 3. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$|f(pr, qs) - f(ps, qr) - \theta(pq, rs)f(p, q)g(r, s)| \leq \varepsilon \quad \forall\, p, q, r, s \in G. \tag{2.15}$$

Then f (or g) is bounded or g satisfies (CDM).

Proof. For the case g is bounded, it is the same reason as the boundedness of Corollary 2. $\square$

Corollary 4. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$|f(pr, qs) - f(ps, qr) - \theta(pq, rs)f(p, q)g(r, s)| \leq \varepsilon \quad \forall\, p, q, r, s \in G, \tag{2.16}$$

and $|f(p, q) - g(p, q)| \leq M$ for all $p, q \in G$ and some constant M.

Then, either g is bounded or f satisfies (CSM). In particular, if g satisfies (CDM), then f and g satisfy (CDM_{fffg}) without above bounded condition M.

Finally, all of the obtained results imply the following result.

Corollary 5. *Let $f, g : G^2 \to \mathbb{R}$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying*

$$|f(pr, qs) - f(ps, qr) - \theta(pq, rs)f(p, q)f(r, s)| \leq \varepsilon \quad \forall\, p, q, r, s \in G.$$

Then f is bounded

Proof. If f is unbounded, from Theorem 5, f satisfies (CDM) and (CSM), simultaneously. It is a contradiction by nonzeroness of function f. Hence the unboundedness of f is false. $\square$

3. Extension to the Banach space

On all of the results obtained in Section 2, the range of function within the Abelian group can be extended to the Banach space. For simplicity, we will only present the extended case of Theorem 1. Given that the other cases are similar, its illustration will be omitted.

Theorem 6. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach space.*

Suppose that $f, g : G^2 \to E$, $\phi : G^2 \to \mathbb{R}_+$ be functions and a function $\theta : G^2 \to \mathbb{R}_k$ be a cocycle satisfying

$$\|f(pr, qs) - f(ps, qr) - \theta(pq, rs)g(p, q)f(r, s)\| \leq \phi(r, s) \quad \forall\, p, q, r, s \in G. \tag{3.1}$$

and $\|f(p, q) - g(p, q)\| \leq M$ for all $p, q \in G$ and some constant M.

For an arbitrary linear multiplicative functional $x^* \in E^*$, then either $x^* \circ g$ is bounded or f satisfies the equation (CDM)

Proof. Assume that (3.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$, hence, for every $x, y \in G$, we have

$$\begin{aligned}\phi(r,s) &\geq \|f(pr,qs) - f(ps,qr) - \theta(pq,rs)g(p,q)f(r,s)\| \\ &= \sup_{\|y^*\|=1} \left|y^*\big(f(pr,qs) - f(ps,qr) - \theta(pq,rs)g(p,q)f(r,s)\big)\right| \\ &\geq \left|x^*\big(f(pr,qs)\big) + x^*\big(f(ps,qr)\big) - x^*\big(\theta(pq,rs)\big)\, x^*\big(g(p,q)\big)\, x^*\big(f(r,s)\big)\right|,\end{aligned}$$

which states that the superpositions $x^* \circ f$ and $x^* \circ g$ yield a solution of inequality (2.1) of Theorem 1. Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 1 shows that the function $x^* \circ f$ solves (CDM).

In other words, bearing the linear multiplicativity of x^* in mind, for all $p, q, r, s \in G$, the difference $\mathcal{D}f(p,q,r,s) : G^2 \to \mathbb{C}$ defined by

$$\mathcal{D}f(p,q,r,s) := f(pr,qs) - f(ps,qr) - \theta(pq,rs)f(p,q)f(r,s)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathcal{D}f(p,q,r,s) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $p, q, r, s \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$\mathcal{D}f(p,q,r,s) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed. □

Remark 1. *i) We obtain the same number of corollaries on the Banach space for all the theorems mentioned in Section 2.*

ii) The domains of all of the obtained results can be extended to the following cases: Abelian group $: G^2$ *is replaced by the real field* $: R^2$ *and* $I^2 = (0,1)^2$.

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