A Note on Hypercyclicity Sets

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Abstract
In this paper, we show that if a continuous linear operator on a separable F-space admits a syndetic hypercyclicity sequence, then the operator is weakly mixing.

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1 Introduction
Let X be a separable F-space and denote by $\mathcal{L}(X)$ the space of all continuous linear operators on X. An operator $T \in \mathcal{L}(X)$ is said to be hypercyclic if there is a vector $x \in X$ such that the orbit $O(x, T) = \{T^n x \mid n \in \mathbb{N}\}$ is dense in X. The density of the orbit implies that there is an increasing sequence $(b_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} T^{b_n} x = x$. Based on this property, a hypercyclicity sequence for operator $T$ is defined as in [2]. In this paper, we focus on the properties
of a hypercyclicity sequence for a hypercyclic operator. More precisely, we will show that if the sequence is syndetic (that is, $\sup_n (b_n - b_{n-1}) < \infty$), then the corresponding operator is weakly mixing. This kind of property has been proved in [5]. Even though our result is basically the same as the one in [5], we mainly focus on the properties of hypercyclicity sequence and we interprete the linear dynamics of $T \in \mathcal{L}(X)$ in terms of the hypercyclicity sequences.

2 Hypercyclicity Sequences

Let $T$ be a hypercyclic operator on a separable $F$-space $X$. Then there is a vector $x \in X$ such that the $T$-orbit

$$O(x, T) = \{T^n x \mid n \in \mathbb{N}\}$$

is dense in $X$. Thus, for each non-empty open subset $U$ of $X$, the set

$$\mathcal{N}(x, U) = \{m \in \mathbb{N} \mid T^m x \in U\}$$

is non-empty. Since $O(x, T)$ is dense in $X$, there is an increasing sequence $(b_n)_{n \in \mathbb{N}}$ of natural numbers such that

$$\lim_{n \to \infty} T^{b_n} x = x.$$ 

Since $T$ is continuous, for $m \in \mathcal{N}(x, U)$, we have

$$\lim_{n \to \infty} T^{m+b_n} x = T^m x \in U$$

In other words, there is a sufficiently large integer $N \geq 0$ such that for all $k \geq 1$

$$m + b_{k+N} \in \mathcal{N}(x, U) \quad \text{and} \quad (1)$$

$$T^{b_{k+N}} T^m x \in U \quad (2)$$

Since $T^m x \in U$, $T^{b_{k+N}} U \cap U \neq \emptyset$. Thus

$$b_{k+N} \in \mathcal{N}(U, U) = \{l \in \mathbb{N} \mid T^l U \cap U \neq \emptyset\}$$

Consider the difference set, defined by

$$\mathcal{N}(x, U) - \mathcal{N}(x, U) = \{l_1 - l_2 \mid l_1 \geq l_2 \text{ and } l_1, l_2 \in \mathcal{N}(x, U)\}$$

Since $m \in \mathcal{N}(x, U)$, for all $k \geq 1$

$$b_{k+N} \in \mathcal{N}(x, U) - \mathcal{N}(x, U)$$
In fact, it is easy to see that if \( x \in X \) is a hypercyclic vector for \( T \), then
\[
N(U, U) = N(x, U) - N(x, U)
\]
By (1), for all \( k \geq 1 \),
\[
b_{k+1+N} - b_{k+N} \in N(x, U) - N(x, U) = N(U, U)
\]
From this analogy and the definition given in [2] and [3], we define a notion of hypercyclic operators as follows:

**Definition 2.1.** Let \( T \in \mathcal{L}(X) \). An increasing sequence \((b_n)_{n \in \mathbb{N}}\) of natural numbers is said to be hypercyclicity sequence for \( T \) if there is a vector \( x \in X \) and integers \( m, N \) such that for each non-empty open subset \( U \) of \( X \)
\[
m + b_{k+N} \in N(x, U) \quad \text{for all } k \geq 1.
\]

**Remark 2.2.** It is clear from the definition that if an operator \( T \in \mathcal{L}(X) \) admits a hypercyclicity sequence then \( T \) is hypercyclic.

**Remark 2.3.** Let \((b_n)_{n \in \mathbb{N}}\) be a hypercyclicity sequence for an operator \( T \in \mathcal{L}(X) \). If the sequence is of order \( m_k \), in other words \( b_k \leq m_k \) for all integers \( k \geq 1 \), then the operator \( T \) is \( m_k \)-hypercyclic. In particular, if \( m_k = k \), then \( T \) is frequently hypercyclic. See [1, 4, 3, 7] and [2] for definitions of frequently hypercyclic operators and \( m_k \)-hypercyclic operators.

### 3 Weak Mixing and Hypercyclicity Sequences

As we have seen in Section 2, the notion of a hypercyclicity sequence is closely related with the known hypercyclic operators such as frequently hypercyclic operators. In this section, we study the relationship between weakly mixing operators and hypercyclicity sequences following the ideas of [5].

By definition, an operator \( T \in \mathcal{L}(X) \) is said to be weakly mixing if the operator \( T \oplus T \) is hypercyclic. Thus by applying the properties of hypercyclic operators we have the followings:

1. There is a vector \( x \oplus y \in X \oplus X \) such that the orbit
\[
O(x \oplus y, T \oplus T) = \{(T \oplus T)^n(x \oplus y) \mid n \in \mathbb{N}\}
\]
is dense in \( X \oplus X \)
2. There is an increasing sequence \((b_n)_{n \in \mathbb{N}}\) such that
\[
\lim_{n \to \infty} (T \oplus T)^{b_n} (x \oplus y) = x \oplus y
\]
\[
\lim_{n \to \infty} T^{b_n} x = x \quad \lim_{n \to \infty} T^{b_n} y = y
\]

3. For any pair \(U, V\) of non-empty open subsets of \(X\), there is an integer \(l \in \mathbb{N}\) such that
\[
(T \oplus T)^l (x \oplus y) \in U \oplus V
\]
In other words,
\[
l \in \mathcal{N}(x, U) \cap \mathcal{N}(y, V).
\]
Thus, \(T\) is a weakly mixing operator on \(X\) if and only if there exist vectors \(x, y\) in \(X\) such that for each pair \(U, V\) of non-empty open subsets of \(X\)
\[
\mathcal{N}(x, U) \cap \mathcal{N}(y, V) \neq \emptyset.
\]
As we obtain the relation (3), we have the following property.

**Proposition 3.1.** Let \((b_n)_{n \in \mathbb{N}}\) be a hypercyclicity sequence for a weakly mixing operator \(T\) on \(X\). Then for any pair of non-empty open subsets \(U\) and \(V\), there is an positive integer \(N\) such that for all \(k \geq N\)
\[
b_{k+1} - b_k \in \mathcal{N}(U, U) \cap \mathcal{N}(V, V)
\]
\[
\square
\]

Conversely, we find a condition for a hypercyclicity sequence for some operator to be weakly mixing. For this, we will use the following equivalent form of weakly mixing operators and the proof can be found in [6, 7].

**Proposition 3.2.** \(T\) is weakly mixing if and only if for any pair of non-empty open subsets \(U\) and \(V\) of \(X\)
\[
\mathcal{N}(U, U) \cap \mathcal{N}(U, V) \neq \emptyset
\]
\[
\square
\]

An increasing sequence \((n_k)_{k \in \mathbb{N}}\) is said to be syndetic if
\[
\sup_k (n_{k+1} - n_k) < \infty
\]
and it is easy to see that if a sequence \((n_k)_{k \in \mathbb{N}}\) is syndetic, then there is a number \(M\) such that
\[
(n_k)_{k \in \mathbb{N}} \cap [n, n + M] \neq \emptyset \quad \text{for all } n \in \mathbb{N}
\]

We now have come to the main result
Theorem 3.3. Let \((b_k)_{k \in \mathbb{N}}\) be a hypercyclicity sequence for a hypercyclic operator \(T \in \mathcal{L}(X)\). If the sequence \((b_k)_{k \in \mathbb{N}}\) is syndetic, then \(T\) is weakly mixing.

Proof. Let \(m_k = b_k - b_{k-1}\). Since the sequence \((b_k)_{k \in \mathbb{N}}\) is syndetic, there is a number \(M\) such that for each \(k\), \(1 \leq m_k \leq M\). Also, \(m_k \in \mathcal{N}(U,U)\) implies that \(T^{m_k}U \cap U \neq \emptyset\). For each \(l \in [1, M]\), let
\[
U_l = U \cap T^{-l}U
\]
Each set \(U_l\) is non-empty open set and there is an element \(u_l \in U_l\) such that \(T^l u_l \in U\). By the topological transitivity for open sets \(U_l\) and \(V\), there is an integer \(k_l\) such that \(T^{k_l}U_k \cap V \neq \emptyset\). Thus for some \(u_l \in U_l\), \(T^{k_l}u_l \in V\). Now,
\[
T^{k_l}T^{-l}T^l u_l \in V
\]
and thus
\[
k_l - l \in \mathcal{N}(U,V), \quad 1 \leq l \leq M.
\]
Since the sequence \((b_k)_{k \in \mathbb{N}}\) is syndetic,
\[
(b_k)_{k \in \mathbb{N}} \cap [k_l - M, k_l] \neq \emptyset.
\]
Thus there is some \(k \in \mathbb{N}\) with \(b_k = k_l - j \in [k_l - M, k_l]\) which implies that
\[
\mathcal{N}(U,U) \cap \mathcal{N}(U,V) \neq \emptyset
\]
By Proposition 3.2, \(T\) is weakly mixing. \(\square\)

References


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