Metric Analysis

Properties and Applications as a Tool for

Smoothing

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Abstract

One of the primary goals of data processing is the problems of smoothing of investigated function values. Different methods and schemes for solving various smoothing problems are developed and in use [1-5]. The new approach smoothing algorithms for functions of one and many variables based on this approach are offered. The schemes based on the metric analysis are also used for problems of smoothing and restoration of functional values [6-10]. It is shown that the metric analysis with high accuracy solves the problem of smoothing and restoration of functions of many variables even when the values of the function are known only in a small number of points.
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1 Introduction

We will remind that the classical schemes are based on the representation of smoothing function $y_{smo}(x)$ in the form of a linear combination:

$$y_{smo}(x) = \sum_{j=1}^{m} c_j \cdot \varphi_j(x),$$

(1)

where $\varphi_j(x), j = 1, ..., m$ is the system of basic functions, $c_j$ are required parameters.

The scheme of representation of smoothing functions in the form of linear combinations of basic functions, including polynomials and splines approximations, basically, can be generalized on function of many variables, but almost all such schemes are efficient only for functions of small number variables.

For functions of big number of variables there are no effective general schemes of smoothing based on the representation (1). There are only rough schemes of smoothing such as local linear approximations which usually require a big amount of initial data. An example of such a scheme is the neural networks that can be used for smoothing of functions of many variables [4, 5]. In this paper it is represented a universal approach that allows effectively solve the smoothing problems for functions of many variables without fixing a priori the functional dependence on the arguments of function. This approach uses only the realized values of function $Y_i, i = 1, ..., n$ at given points $X_i, ..., X_n$ including the presence of chaotic errors in values $Y_i, i = 1, ..., n$.

2 The smoothing schemes based on the metric analysis

Let's consider the problem of smoothing and restoration of functional dependence $Y = F(X_1, ..., X_m) = F(X)$ in the presence of chaotic deviations from exact values. The values of function $Y_i = F(X_i), i = 1, ..., n$ are known with errors at points $X_i = (X_{i1}, ..., X_{im})^T, i = 1, ..., n$.

Assume the matrix of metric uncertainty $W$ is not singular [11].

For any point $\bar{X}$ we are looking for the smoothing value $Y^*$ in representation

$$Y^* = \sum_{i=1}^{n} z_i \cdot Y_i = (\bar{z}, \bar{Y}),$$

(2)

where the vector of weights $\bar{z}$ is the solution of the following problem on a minimum of the total uncertainty:

$$(W\bar{z}, \bar{z}) + \alpha \cdot (K_y \bar{z}, \bar{z}) - \min \bar{z},$$

(3)

where $\alpha$ is the smoothing parameter.
While the expression \((W \bar{z}, \bar{z})\) is responsible for metric uncertainty, the expression \((K_\parallel \bar{z}, \bar{z})\) is responsible for stochastic uncertainty. The problem (3) can be solved using the Lagrange's function. The smoothed value at the point \(\bar{X}^*\) is given by the equality:

\[
Y_{sm}^* = ((W + \alpha \cdot K_\parallel)^{-1} \bar{1}, \bar{Y}) / ((W + \alpha \cdot K_\parallel)^{-1} \bar{1}, \bar{1}).
\]  
(4)

When \(\alpha \to + \infty\) the smoothed value \(Y_{sm}^*\) at any point \(\bar{X}^*\) is:

\[
Y_{sm}^* = (K_\parallel)^{-1} \bar{1}, \bar{Y}) / (K_\parallel)^{-1} \bar{1}, \bar{1}).
\]  
(5)

**Theorem 1.** At points \(\bar{X}^* = \bar{X}_t\) when \(\alpha \to +0\) the solution of problem (3) goes to the value of function \(Y_t = F(\bar{X}_t)\).

**Proof.**

The Lagrange’s function for the problem (4) has the form

\[
L(\bar{z}, u) = \frac{1}{2} \cdot ((W + \alpha \cdot K_\parallel)\bar{z}, \bar{z}) + u \cdot (1 - (\bar{z}, \bar{1})),
\]

\[
\nabla \bar{z} L = (W + \alpha \cdot K_\parallel)\bar{z} - u \cdot \bar{1} = 0.
\]  
(6)

Since the \(i\) - th row and the \(j\) - th column of the matrix \(W\) consists of zeros we have the system of equations

\[
\alpha \cdot \sum_{j=1}^{n} (K_\parallel)_{ij} z_j = u \Rightarrow z_i = \frac{u}{\alpha (K_\parallel)_{ij}} \cdot \left( \frac{(K_\parallel)_{ii}}{(K_\parallel)_{ii}} \right).
\]  
(7)

where we have introduced the notations

\[
\bar{z}_i = (z_1, ..., z_{i-1}, z_{i+1}, ..., z_n)^T, \bar{K}_i = ((K_\parallel)_{i1}, ..., (K_\parallel)_{ii-1}, (K_\parallel)_{ii+1}, ..., (K_\parallel)_{in})^T.
\]

The \(r\) - th equation of the system (6), \(r \neq i\), can be rewritten as

\[
\sum_{j=1, j \neq i}^{n} W_{rj} + \alpha \cdot (K_\parallel)_{rj} - \alpha \cdot \frac{(K_\parallel)_{ri}}{(K_\parallel)_{ii}} \cdot (K_\parallel)_{ij} z_j = u \cdot \left( 1 - \frac{(K_\parallel)_{ri}}{(K_\parallel)_{ii}} \right).
\]

We denote by \(M_\alpha\) a matrix which size is \((n-1) \times (n-1)\) with elements

\[
(M_\alpha)_{rj} = W_{rj} + \alpha \cdot (K_\parallel)_{rj} - \alpha \cdot \frac{(K_\parallel)_{ri}}{(K_\parallel)_{ii}} \cdot (K_\parallel)_{ij}, r, j = 1, ... i - 1, i + 1, ..., n. \]  
(8)

We denote by \(\vec{f}\) the column vector which size is \((n-1) \times 1\) with elements
Then we obtain
\[ \vec{Z}_t = u \cdot M^{-1}_\alpha \hat{f} \]  
and from (7)
\[ z_i = u \cdot \left( \frac{1}{\alpha(K_{ii})} - \frac{(\vec{K}_{ii} M^{-1}_\alpha \hat{f})}{(K_{ii})} \right) \]  
We obtain from the condition \( (\vec{z}, \vec{1}) = 1 \)
\[ u(\alpha) = \frac{\alpha(K_{ii})}{1 + \alpha(K_{ii})u'(M^{-1}_\alpha \hat{f}, \vec{1}) - \alpha(\vec{K}_i, M^{-1}_\alpha \hat{f})} \]  
Substituting (11) into (9) and (10) we obtain
\[ \vec{Z}_t = \frac{\alpha(K_{ii})}{1 + \alpha(K_{ii})u'(M^{-1}_\alpha \hat{f}, \vec{1}) - \alpha(\vec{K}_i, M^{-1}_\alpha \hat{f})} \]  
\[ z_i = \frac{1 - \alpha(\vec{K}_i, M^{-1}_\alpha \hat{f})}{1 + \alpha(K_{ii})u'(M^{-1}_\alpha \hat{f}, \vec{1}) - \alpha(\vec{K}_i, M^{-1}_\alpha \hat{f})} \]  
We denote by \( \vec{W} \) the matrix, which is obtained from the matrix \( W \) by deleting \( j \) - th column and the \( i \) - th row (which consist of zeros).
Then from (8) it follows that
\[ M_\alpha \to \vec{W} \ \text{when} \ \alpha \to 0 \]  
and we obtain from (12) and (13)
\[ \vec{Z}_t \to 0, \ z_i \to 1 \ \text{when} \ \alpha \to 0. \]
We obtain also from (1) that \( \vec{X}^* = \vec{X}_t \)
\[ Y^* \to Y_t \ \text{when} \ \alpha \to 0. \]
The theorem is proved.

3 The metrical analysis smoothing scheme by means of eigen values and eigen vectors of metrical uncertainty matrix

The problem (3) can be solved also by means of eigen values and eigen vectors of matrix \( W \). Let \( \lambda_1, ..., \lambda_n \) be the eigen values, and \( \vec{\phi}_1, ..., \vec{\phi}_n \) corresponding to them orthonormal system eigen vectors of the matrix \( W \).
We decompose the vector of weights $\mathbf{z}$ in this system

$$\mathbf{z} = \sum_{i=1}^{n} c_i \cdot \mathbf{\phi}_i.$$

Let's denote by $F = (\mathbf{\phi}_1, ..., \mathbf{\phi}_n)$ the matrix which columns are the eigen vectors of matrix $W$.

Then

$$\mathbf{z} = F \mathbf{c}, \quad \mathbf{c} = (c_1, ..., c_n)^T. \quad (14)$$

For the total uncertainty we have:

$$(W \mathbf{z}, \mathbf{z}) + \alpha \cdot (K_{\mathbf{z}} \mathbf{z}, \mathbf{z}) = \sum_{i=1}^{n} \lambda_i \cdot c_i^2 + \alpha \cdot \sum_{i,j=1}^{n} c_i \cdot c_j \cdot (K_{\mathbf{z}} \mathbf{\phi}_i, \mathbf{\phi}_j)$$

$$= (D \mathbf{c}, \mathbf{c}) + \alpha \cdot (B \mathbf{c}, \mathbf{c}),$$

where

$$D = \text{diag}(\lambda_1, ..., \lambda_n), \quad B_{ij} = (K_{\mathbf{z}} \mathbf{\phi}_i, \mathbf{\phi}_j), \quad i, j = 1, ..., n,$$

$$(\mathbf{z}, \mathbf{1}) = \sum_{i=1}^{n} c_i \cdot (\mathbf{\phi}_i, \mathbf{1}) = (\mathbf{c}, \mathbf{\phi}), \mathbf{\phi} = ((\phi_1, \mathbf{1}), ..., (\phi_n, \mathbf{1})).$$

From problem (3) for vector $\mathbf{z}$ we pass to the following problem for vector $\mathbf{c}$

$$(D \mathbf{c}, \mathbf{c}) + \alpha \cdot (B \mathbf{c}, \mathbf{c}) - \min \mathbf{c}$$

$$= (\mathbf{c}, \mathbf{\phi}) = 1. \quad (15)$$

We have:

$$L(\mathbf{c}, u) = \frac{1}{2} \cdot ((D + \alpha \cdot B) \mathbf{c}, \mathbf{c}) + u \cdot (1 - (\mathbf{c}, \mathbf{\phi})), \quad \nabla_{\mathbf{c}} L = (D + \alpha \cdot B) \mathbf{c} - u = 0 \Rightarrow \mathbf{c} = u \cdot (D + \alpha \cdot B)^{-1} \mathbf{\phi},$$

$$(\mathbf{c}, \mathbf{\phi}) = 1 \Rightarrow u = \frac{1}{((D + \alpha \cdot B)^{-1} \mathbf{\phi}, \mathbf{\phi})}.$$  

From here we get the required vector $\mathbf{c}$:

$$\mathbf{c} = \frac{(D + \alpha \cdot B)^{-1} \mathbf{\phi}}{((D + \alpha \cdot B)^{-1} \mathbf{\phi}, \mathbf{\phi})}. \quad (16)$$

From (14) we get the vector of weights $\mathbf{z}$.
\[ \hat{z} = F \frac{(D+a\cdot B)^{-1}\phi}{(D+a\cdot B)^{-1}\phi,\phi}, \]  
(17)

and we get the smoothed value

\[ Y_{\text{smo}}^* = \frac{(F(D+a\cdot B)^{-1}\phi,\bar{Y})}{((D+a\cdot B)^{-1}\phi,\phi)}. \]  
(18)

**Remark.** It follows from the proof of the theorem 1 that the smoothed values at points \( \vec{x}_i \) are defined by formulas (12) and (13) (which define the vector of weights \( \vec{z} \)).

Let's show the results of smoothing on concrete examples.

**Exemple 1.**
Consider the function \( f(r) = 2 + \cos(r), \ r = \sqrt{x^2 + y^2} \) given on the plane \( (x, y) \). The function is defined on an irregular grid on the plane \( (x, y) \). The values of function are generated with relative errors 2-8%. On figures 1, 2 are presented surfaces of initial and restored functions respectively.

**Exemple 2.**
Let's consider the function that corresponds to the energy field distribution in a nuclear reactor. On figures 3-5 are presented the surfaces of initial function, function with noise and restored function respectively. The values of function with noise had a relative error of 8%, while the accuracy of restored function remains within 1%.

4 **Conclusion**

In the current paper were presented new schemes and algorithms for smoothing and restoration of functional values based on the metric analysis. These schemes and algorithms have shown high accuracy of smoothing and restoration of functions of one and many variables.
Figure 1. Surface of exact values

Figure 2. The restored surface
Figure 3. Surface of exact values

Figure 4. Surface with noise
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Figure 5. The restored surface

References


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