Some Symmetric Identities for $h$-Extension of $q$-Euler Polynomials under Third Dihedral Group $D_3$

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Abstract

In this paper, we give some new symmetric identities for $h$-extension of $q$-Euler polynomials under third Dihedral group $D_3$. 
1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $| \cdot |_p$ be the normalized $p$-adic absolute value with $|p|_p = \frac{1}{p}$ and let $q$ be an indeterminate in $\mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. The $q$-extension of number $x$ is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \to 1} [x]_q = x$.

The $q$-binomial coefficient is defined by

\[
\binom{x}{n}_q = \frac{[x][x-1] \cdots [x-n+1]_q}{[n]_q!}, \text{ where } [n]_q! = [n][n-1]_q \cdots [2]_q[1]_q.
\]

(1.1)

Let $f$ be a continuous function on $\mathbb{Z}_p$. Then the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [5-15]).}
\]

(1.2)

Thus, by (1.2), we get

\[
I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-l-1}f(l), \text{ (see [10,11]),}
\]

(1.3)

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

For $h \in \mathbb{N}$, the expansion of $q$-Euler polynomials are considered by Kim to be

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^\sum_{i=1}^{r}(h-l)y_i e^{[x+y_1+\cdots+y_r]_q t} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r)
\]

\[
=2^r \sum_{m_1,\ldots,m_r=0}^{\infty} q^\sum_{i=1}^{r}(h-l)m_i (-1)^{\sum_{i=1}^{l} m_i} e^{[x+m_1+\cdots+m_r]_q t}
\]

\[
=2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} q^{(h-r)m} (-1)^m e^{[m+x]_q t}
\]

(1.4)

\[
= \sum_{n=0}^{\infty} E^{(h,r)}_{n,q}(x) \frac{t^n}{n!}, \text{ (see [6,12]).}
\]
From (1.4), we have

\[ E_{n,q}^{(h,r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r (h-l) y_i} [x + y_1 + \cdots + y_r]_q^n d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \]

\[ = 2^r \sum_{m=0}^\infty \binom{m + r - 1}{m} \left(-q^{h-r}\right)^m [x + m]_q^n \]

\[ = \frac{2^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} \left(-q^x\right)^l \left(-q^{h-r+l}\right), \quad \text{where} \quad (b : q)_n = (1-b)(1-bq)\cdots(1-bq^{n-1}). \]

In the special case, \( r = 1 \), we have

\[ E_{n,q}^{(h,1)}(x) = \int_{\mathbb{Z}_p} q^{(h-1)y} [x + y]_q^n d\mu_{-1}(y) \]

\[ = 2 \sum_{m=0}^\infty (-1)^m q^{m(h-1)} [x + m]_q^n \]

\[ = \frac{2}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} \frac{(-1)^l q^x}{1 + q^{h+l-1}}, \]

which is called the \( h \)-extension of \( q \)-Euler polynomials.

Note that \( \lim_{q \to 1} E_{n,q}^{(h,1)}(x) = E_n(x) \), where \( E_n(x) \) are ordinary Euler polynomials which are defined by the generating function to be

\[ \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^\infty E_n(x), \quad \text{where} \quad \text{see [1-22].} \]

In this paper, we give new symmetric identities for the \( h \)-extension of \( q \)-Euler polynomials under third Dihedral group \( D_3 \) which are derive from the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \).

2. Symmetric identities under \( D_3 \)

Let \( w_1, w_2, w_3 \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( w_3 \equiv 1 \pmod{2} \). Then, we have

\[ \int_{\mathbb{Z}_p} q^{(h-1) w_2 w_3 y} e^{w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 + w_1 w_2} q^t d\mu_{-1}(y) \]

\[ = \lim_{N \to \infty} \sum_{y=0}^{p^N-1} q^{(h-1) w_2 w_3 y} e^{w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 + w_1 w_2} q^t (-1)^y \]

\[ = \lim_{N \to \infty} \sum_{y=0}^{p^N-1} \sum_{k=0}^{w_1-1} q^{(h-1)(k+w_1 y) w_2 w_3} (-1)^{k+y} e^{w_2 w_3 (k+w_1 y) + w_1 w_2 w_3 x + w_1 w_3 + w_1 w_2} q^t. \]

(2.1)
From (2.1), we can derive the following equation (2.2):

\[
\sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} q^{(h-1)i+q(w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j) q t} d\mu_{-1}(y) 
\times \int_{\mathbb{Z}_p} q^{(h-1) w_2 w_3 y} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j] q t} d\mu_{-1}(y) 
= \lim_{N \to \infty} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \sum_{k=0}^{w_3-1} \sum_{y=0}^{w_2-1} (-1)^{i+j+k} q^{(h-1)(w_1 w_2 j + w_2 w_3 k + w_1 w_2 y)} 
\times e^{[w_2 w_3 (k+w_1 y) + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j] q t}.
\]

As this expression is invariant under any permutation \(\sigma \in D_3\), we have the following theorem.

**Theorem 2.1.** Let \(w_1, w_2, w_3\) be odd natural numbers. Then the following expressions

\[
\sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} q^{(h-1)(i w_\sigma(1) w_\sigma(3) + j w_\sigma(1) w_\sigma(2))} 
\times \int_{\mathbb{Z}_p} q^{(h-1) w_\sigma(2) w_\sigma(3) y} e^{[w_\sigma(2) w_\sigma(3) y + w_\sigma(1) w_\sigma(2) w_\sigma(3) x + w_\sigma(1) w_\sigma(3) i + w_\sigma(1) w_\sigma(2) j] q t} d\mu_{-1}(y)
\]

are the same for any \(\sigma \in D_3\).

By (1.6), we get

\[
\int_{\mathbb{Z}_p} q^{(h-1) w_2 w_3 y} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j] q t} d\mu_{-1}(y) 
= \sum_{n=0}^{\infty} [w_2 w_3]_q^n E_{n, q = w_2 w_3}^{(h, 1)} \left( w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right) q^n t^n n!.
\]

Therefore, by Theorem 2.1 and (2.3), we obtain the following theorem.

**Theorem 2.2.** Let \(w_1, w_2, w_3\) be odd natural numbers. Then the following expressions

\[
[w_\sigma(2) w_\sigma(3)]_q^n \sum_{i=0}^{w_\sigma(2)-1} \sum_{j=0}^{w_\sigma(3)-1} (-1)^{i+j} q^{(h-1)(i w_\sigma(1) w_\sigma(3) + j w_\sigma(1) w_\sigma(3))} 
\times E_{n, q = w_\sigma(2) w_\sigma(3)}^{(h, 1)} \left( w_\sigma(1) x + \frac{w_\sigma(1)}{w_\sigma(2)} i + \frac{w_\sigma(1)}{w_\sigma(2)} j \right)
\]

are the same for any \(\sigma \in D_3\) and \(n \in \mathbb{N} \cup \{0\}\).
From (1.6), we have
\[
\int_X q^{(h-1)w_1 w_3 y} \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]^n d\mu_{-1}(y)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{[w_1]_q}{[w_2 w_3]_q} \right)^{n-k} [w_3 i + w_2 j]^{n-k} \int_X q^{(h-1)w_2 w_3 y} [y + w_1 x]^k_{q^{w_2 w_3}} d\mu_{-1}(y)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{[w_1]_q}{[w_2 w_3]_q} \right)^{n-k} [w_3 i + w_2 j]^{n-k} E_{k,q^{w_2 w_3}}^{(h,1)}(w_1 x).
\tag{2.4}
\]

By (2.4), we get
\[
[w_2 w_3]_q^{n-k} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} q^{(h-1)(i w_2 w_3 + j w_1 w_2)} \times \int_X q^{(h-1)w_2 w_3 y} \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]^n_{q^{w_2 w_3}} d\mu_{-1}(y)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} [w_2 w_3]_q^k [w_1]_q^{n-k} E_{k,q^{w_2 w_3}}^{(h,1)}(w_1 x) T_{n,k,q^{w_1}}^{(h)}(w_2, w_3),
\]
where
\[
T_{n,k,q}^{(h)}(w_1, w_2) = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} q^{(h-1)(i w_2 w_3 + j w_1 w_2)} [w_2 i + w_1 j]^{n-k}.
\tag{2.6}
\]

Therefore, by (2.5) and (2.6), we obtain the following theorem.

**Theorem 2.3.** For \( w_1, w_2, w_3 \in \mathbb{N} \) with \( w_1 \equiv 1 \) (mod 2), \( w_2 \equiv 1 \) (mod 2), \( w_3 \equiv 1 \) (mod 2), and \( n \in \mathbb{N} \cup \{0\} \), the following expressions
\[
\sum_{k=0}^{n} \binom{n}{k} [w_{\sigma(2)} w_{\sigma(3)}]_q^k [w_{\sigma(1)}]_q^{n-k} E_{k,q^{w_{\sigma(2)} w_{\sigma(3)}}}^{(h,1)}(w_{\sigma(1)} x) T_{n,k,q^{w_{\sigma(1)}}}^{(h)}(w_{\sigma(2)}, w_{\sigma(3)})
\]
are the same for any \( \sigma \in D_3 \).

**References**


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