Uniqueness of a Meromorphic Function and its Differential Polynomial

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Abstract

In this paper, we investigate the problem of uniqueness of a non-constant meromorphic function \( f \) and its differential polynomial \( Q[f] \) when they share two distinct, non-zero, small meromorphic function \( IM^* \), where we have taken the conditions \( N(r, f) = S(r, f) \) and \( N(r, 0, f) < \lambda T(r, f), \ \lambda \in (0, 1) \).

Mathematics Subject Classification: 30D35, 30D30

Keywords: Meromorphic function, Shared values, Linear combination of Derivatives, Differential Polynomial

I. INTRODUCTION

Let \( \mathbb{C}^* \) denote the extended complex plane. We say that two meromorphic function \( f \) and \( g \) share the value \( a \in \mathbb{C}^* \), \( CM(IM) \) provided that \( f(z) = a \) if and only if \( g(z) = a \), counting multiplicity (ignoring multiplicity). It is assumed that the reader is familiar with the usual notations and fundamental results of Nevanlinna theory of meromorphic function see e.g. [3]. In the sequel, a meromorphic function \( a(z) \) is called a small function of \( f \) if and only if \( T[r, a(z)] = o(T[r, f]) \) as \( r \to \infty \) possibly outside a set of \( r \) of finite linear measure.
In 1996, Dhar [2] studied some results about sharing of a function and its linear differential polynomials and proved the following result:

**THEOREM A**: Let $f$ be a non-constant entire function and
\[
L[f] = \sum_{j=1}^{k} a_j f',
\]
share two finite values CM, then $f \equiv L[f]$.

We know that if $f$ is a non-constant meromorphic function then $f$ and $L[f]$ cannot share two finite values IM, otherwise $f \neq L[f]$ as shown by the following example given by Gundersen [1]:

**EXAMPLE**: Let
\[
f(z) = \frac{2A}{1 - Be^{-2z}}, \quad A \neq 0, B \neq 0,
\]
then $f$ and $f'$ share 0 and 1 DM (different multiplicities) and $f \neq f'$. In the example, one shared value is zero. Therefore, there arises the following question:

**II. QUESTION**

If $f(z)$ is a non-constant meromorphic function which shares two finite, non-zero small meromorphic functions IM with $L[f]$, a linear differential polynomial in $f$, then what about their uniqueness?

**III. DEFINITIONS**

**DEFINITION 1**: Let $p_0, p_1, \ldots, p_k$ be non-negative integers. We call
\[
M[f] = f^{p_0} (f')^{p_1} \ldots (f^{(k)})^{p_k},
\]
a monomial in $f$ with $d_M = p_0 + p_1 + \ldots + p_k$, its degree. Further let $M_1[f], M_2[f], \ldots, M_n[f]$ denote monomials in $f$ and $a_1, a_2, \ldots, a_n$ small meromorphic functions such that $T(r,a_j) = S(r,f), 1 \leq j \leq n$.

Then
\[
Q[f] = \sum_{j=1}^{n} a_j M_j[f],
\]
is called a differential polynomial in $f$ of degree
\[
d_Q = \text{Max} \, d_M.
\]
The Nevanlinna Deficiency is defined by
Uniqueness of a meromorphic function

$\delta(a, f) = 1 - \lim_{r \to \infty} \frac{N(r, a, f)}{T(r, f)} = \lim_{r \to \infty} \frac{m(r, a, f)}{T(r, f)}$. 

For statement of our results, we need the following definitions:

**DEFINITION 2:**
A value $a$ is said to be shared by $f$ and $g$ CM* if

$\overline{N}(r, a, f) - \overline{N}_E(r, a, f) = S(r, f)$

where $\overline{N}_E(r, a, f)$ is the counting function (counted only once) of those $a$- points of $f$, where $a$ is taken by $f$ and $g$ with same multiplicity.

**DEFINITION 3:**
A value $a$ is said to be shared by $f$ and $g$ IM* if

$\overline{N}(r, a, f) - \overline{N}_I(r, a, f) = S(r, f)$

where $\overline{N}_I(r, a, f)$ is the counting function (counted only once) of those $a$- points of $f$, where $a$ is taken by $f$ and $g$ with ignoring multiplicity.

In the above definitions, the constant value $a$ can be replaced by a small meromorphic function $a(z)$.

**IV. MAIN RESULTS**

**THEOREM 1:** Let $f$ be a non-constant meromorphic function such that $N(r, f) = S(r, f) \& N(r, 0, f) \leq \lambda T(r, f), \lambda \in (0,1)$

Let

$$Q[f] = \sum_{j=1}^{n} a_jM_j[f],$$

be a non-constant differential polynomial which shares two distinct, finite, non-zero small meromorphic functions $a, b$ IM* with $f^{d_Q}$, where $d_Q$ is the degree of $Q[f]$, then

$$f^{d_Q} \equiv Q[f].$$

For the proof of the theorem, we need the following lemma due to Doeringer [4]:

**LEMMA:** Suppose that $f$ is meromorphic and $f^P = Q$, where $P$ and $Q$ are differential polynomials in $f$ and the degree of $Q$ is at most $n$.

Then
\[ m(r, P) = S(r, f), r \to \infty \]

**Proof of Theorem 1:** Suppose \( f^{d_0} \neq Q[f] \). Let \( a, b \) be two distinct, finite non-zero small meromorphic functions of \( f \) such that \( f^{d_0} \) and \( Q[f] \) share \( a, b \) then by using \( N(r, f) = S(r, f) \), above lemma and Nevanlinna’s Second Fundamental Theorem, we have

\[
[d_0 + o(1)]T(r, f) = T(r, f^{d_0})
\]

\[
\leq \overline{N}(r, a, f^{d_0}) + \overline{N}(r, b, f^{d_0}) + \overline{N}(r, f)
\]

\[
\leq \overline{N}(r, \frac{Q}{f^{d_0}}, 1)
\]

\[
\leq T(r, \frac{Q}{f^{d_0}})
\]

\[
\leq N(r, \frac{Q}{f^{d_0}}) + S(r, f)
\]

\[
\leq N(r, \frac{1}{f^{d_0}}) + S(r, f)
\]

\[
\leq \lambda d_0 T(r, f) + S(r, f)
\]

which is a contradiction as \( \lambda < 1 \). Thus we have \( f^{d_0} \equiv Q[f] \).

**Remark 1:** The number 2 is best possible in the above Theorem 1 for consider the following:

**Example:** Let \( f(z) = \sin z \) and \( Q[f] = f^2 + f^3 + f^4 \). Then \( f^{d_0} \) and \( Q[f] \) share \( \frac{1}{2} \).

**Remark 2:** If we replace \( Q[f] \) by \( L[f] \) (as defined in Theorem A) in Theorem 1 then \( d_0 = 1 \) and we get the following result which is answer to above Question:

**Theorem 2:** Let \( f \) be a non-constant meromorphic function such that \( N(r, f) = S(r, f) \) \& \( N(r, 0, f) \leq \lambda T(r, f), \lambda \in (0, 1) \),

which shares two distinct, finite, non-zero small meromorphic functions \( a, b \) IM* with \( L[f] \), then

\[ f \equiv L[f]. \]
**REMARK 3:** If we replace \( f \) by an entire function in Theorem 2, we get the following result which is an improvement of Theorem A:

**THEOREM 3:** Let \( f \) be a non-constant entire function and \( L[f] \) share two distinct, finite, non-zero small meromorphic functions \( a, b \in \mathbb{M}^* \), then

\[
f = L[f],
\]

provided that \( N(r,0,f) \leq \lambda T(r,f), \lambda \in (0,1) \).

**REFERENCES**


**Received:** August 3, 2014