Classification of Sturm-Liouville Problems

at Infinity

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Abstract

Abstract. We determine the values of $k$ and $p$ such that the Sturm-Liouville differential operator $\tau u = -\frac{d^2 u}{dx^2} + kx^p u$ is in limit point case or limit circle case at infinity. In particular it is shown that $\tau$ is in the limit point case when (i) For $p = 2$ and $\forall k$ (ii) For $\forall p$ and $k = 0$ (iii) For all $p$ and $k > 0$ (iv) For $0 \leq p \leq 2$ and $k < 0$ (v) For $p < 0$ and $k < 0$. $\tau$ is in the limit circle case when (i) For $p > 2$ and $k < 0$.

Keywords: limit point case, limit circle case

1 Introduction

We study self-adjoint operators generated by Sturm-Liouville differential expressions
\[ \tau u(x) = \frac{1}{r(x)} \{ -(pu')'(x) + q(x)u(x) \} \text{ in } (a, b), \quad -\infty \leq a < b \leq \infty \]

in the Hilbert space \( L_2(a, b; r) \) with inner product
\[ \langle u, v \rangle = \int_a^b u(x) v(x) r(x) \, dx. \]

We require the following minimal assumptions on the coefficients of \( \tau \):
- \( p, q, r \) are real-valued measurable functions on \((a, b)\).
- \( p(x), r(x) > 0 \) almost everywhere in \((a, b)\).
- \( \frac{1}{p}, q, r \) are locally integrable in \((a, b)\).

We will consider here the case when \( p(x) = r(x) = 1 \). Then \( \tau \) defined above becomes
\[ \tau u(x) = -u''(x) + q(x)u(x) \text{ in } (a, b), -\infty \leq a < b \leq \infty \quad \text{.................[1]} \]

The above assumptions reduce in this case to:
- \( q \) is real-valued measurable functions on \((a, b)\).
- \( q \) is locally integrable in \((a, b)\).

**Definition 1.1** ([10]). We say that \( \tau \) is regular at \( a \), if \( a > -\infty \) and above assumptions hold in \([a, b)\) instead of \((a, b)\). \( \tau \) is called regular if it is regular at \( a \) and at \( b \). \( \tau \) is said to be singular at \( a \) (resp. \( b \)) if it is not regular at \( a \) (resp. \( b \)). \( \tau \) is said to be singular if it is singular at \( a \) or at \( b \).

**Definition 1.2** ([10]). We say that \( \tau \) is in the limit circle case (l.c.c.) at \( b \), if for every \( \lambda \in \mathbb{C} \) all solutions of \((\tau - \lambda)u = 0\) lie right in \( L_2(a, b; r) \). \( \tau \) is in the limit point case (l.p.c.) at \( b \), if for every \( \lambda \in \mathbb{C} \) there is at least one solution of \((\tau - \lambda)u = 0\), which does not lie right in \( L_2(a, b; r) \).

### 2 Criteria for Determining the Limit Point and Limit Circle case for Sturm-Liouville Differential Operator

There are number of limit point-limit circle criteria in the literature. Here we just mention the most important one for the special case [1].

**Theorem 2.1** ([10])
Let \( p(x) = r(x) = 1 \) in \((a, \infty)\) and assume that for some \( c \in (a, \infty) \) and \( k \geq 0 \)
\[ q(x) \geq -kx^2 \text{ for } x \geq c \]
Then \( \tau \) is in the limit point case at \( \infty \).

**Theorem 2.2** ([7])
Let \( q \) be a twice continuously differentiable real-valued function on \((0, \infty)\) and suppose that \( q(x) \to -\infty \) as \( x \to \infty \). Suppose further that
\[
\int_{c}^{\infty} \left( \left( \frac{(-q)^{1/2}}{(-q)^{3/2}} \right)^{'} \right) (-q)^{-1/4} \, dx < \infty
\]
for some \( c \). Then \( q \) is in the Limit point case at \( \infty \) if and only if
\[
\int_{c}^{\infty} (-q(x))^{-1/2} \, dx = \infty.
\]

3. Classification of Sturm-Liouville problems at Infinity for function \( kx^p \) for various values of \( p \) and \( k \) in terms of Limit Point and Limit Circle case

**Proposition 3.1**

For \( p = 2 \) and \( k \neq 0 \), \( \tau = -\frac{d^2}{dx^2} + kx^p \) is in the limit point case at \( \infty \).

**Proof.** Let \( q(x) = kx^p \)

For \( p = 2 \) and \( k > 0 \),

\[
\therefore kx^2 = kx^2 \geq -kx^2, \quad \forall x
\]

\[
\therefore q(x) \geq -kx^2
\]

By theorem 2.1, \( \tau \) is in the limit point case at \( \infty \).

For \( p = 2 \) and \( k = 0 \)

\[
q(x) = 0 \geq 0 \cdot x^2
\]

By theorem 2.1, \( \tau \) is in the limit point case at \( \infty \).

For \( p = 2 \) and \( k < 0 \)

\[
\therefore kx^2 = -(-kx^2)
\]

\[
q(x) = kx^2 \geq -k'x^2, \text{ where } k' = -k > 0
\]

By theorem 2.1, \( \tau \) is in the limit point case at \( \infty \).

**Proposition 3.2**

For \( \forall p \) and \( k = 0 \), \( \tau = -\frac{d^2}{dx^2} + kx^p \) is in the limit point case at \( \infty \).

**Proof.** Let \( q(x) = kx^p \)

For \( \forall p \) and \( k = 0 \),

\[
q(x) = 0 \geq -kx^2, \quad \forall x
\]

By theorem 2.1, \( \tau \) is in the limit point case at \( \infty \).

**Proposition 3.3**

For all \( p \) and \( k > 0 \), \( \tau = -\frac{d^2}{dx^2} + kx^p \) is in the limit point case at \( \infty \).

**Proof.** Let \( q(x) = kx^p \)
\[ kx^p \geq 0 \geq -x^2 \quad \text{for all } \ x > 0 \]

By theorem 2.1, \( \tau \) is in the limit point case at \( \infty \).

**Proposition 3.4**

For \( 0 \leq p \leq 2 \) and \( k < 0 \), \( \tau = -\frac{d^2}{dx^2} + kx^p \) is in the limit point case at \( \infty \).

**Proof.** Since \( 0 \leq p \leq 2 \)

\[
\frac{p}{2} \leq 1 \Rightarrow \frac{2}{p} \geq 1
\]

\[ x^{2/p} \geq x, \quad x \geq 1 \]

\[ x^2 \geq x^p, \quad x \geq 1 \]

\[ kx^2 \leq kx^p, \quad x \geq 1 \]

Put \( k' = -k \)

\[ -k'x^2 \leq kx^p, \quad k' > 0, x \geq 1 \]

By theorem 2.1, \( \tau \) is in the limit point case at \( \infty \).

**Proposition 3.5**

For \( p < 0 \) and \( k < 0 \), \( \tau = -\frac{d^2}{dx^2} + kx^p \) is in the limit point case at \( \infty \).

**Proof.** If \( p < 0 \), then

\[ kx^p = \frac{k}{x^m}, \quad \text{where } m = -p. \]

Now \( \frac{k}{x^m} \geq -cx^2, \ x \geq 1 \) for some \( c > 0 \).

\[ \Rightarrow k \geq -cx^{m+2}, \ x \geq 1. \]

Choose \( c = -k. \) Then \( c > 0 \) and \( x \geq 1 \)

\[ x^{m+2} \geq 1 \]

\[ \Rightarrow -cx^{m+2} \leq -c = k, \ x \geq 1 \]

\[ \Rightarrow \frac{k}{x^m} = kx^p \geq -cx^2, \ x \geq 1. \]

By theorem 2.1, \( \tau \) is in the limit point case at \( \infty \).

**Theorem 3.6**

Let \( k < 0 \) and \( p > 0 \). \( \tau = -\frac{d^2}{dx^2} + kx^p \) is in the limit point case at \( \infty \) if and only if \( p \leq 2 \).

**Proof.** Let \( q(x) = kx^p \) and \( k < 0 \). Then \( q(x) \to -\infty \) as \( x \to \infty \) for \( p > 0 \).

Now for some \( c > 0 \)

\[
\int_c^\infty \left( \frac{[(-q)^{1/2}]}{(-q)^{3/2}} \right)'(-q)^{-1/4}dx = \int_c^\infty \left( \frac{[(-kx^p)^{1/2}]}{(-kx^p)^{3/2}} \right)'(-kx^p)^{-1/4}dx
\]
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\[
\begin{align*}
&= (-k)^{-5/4} \int_c^\infty \left( \frac{p}{x^{2-3p}} \right)' x^{-\frac{p}{4}} dx \\
&= \frac{p}{2} (-k)^{-5/4} \int_c^\infty \left( \frac{p}{x^{2-1-3p}} \right)' x^{-\frac{p}{4}} dx \\
&= \frac{p}{2} (-k)^{-5/4} \int_c^\infty (x^{-p-1})' x^{-\frac{p}{4}} dx \\
&= \frac{p}{2} (k)^{-5/4} \int_c^\infty (x^{-p-1})' x^{-\frac{p}{4}} dx \\
&= (p + 1) \frac{p}{2} (k)^{-5/4} \int_c^\infty x^{-p-2} x^{-\frac{p}{4}} dx \\
&= (p + 1) \frac{p}{2} (k)^{-5/4} \int_c^\infty \frac{1}{x^{\frac{5p}{4} + 2}} dx
\end{align*}
\]

Now

\[
\int_c^\infty \frac{1}{x^{\frac{5p}{4} + 2}} dx < \infty
\]

\[
\Leftrightarrow \frac{5}{4} p + 2 > 1
\]

\[
\Leftrightarrow \frac{5}{4} p > -1
\]

\[
\Leftrightarrow 5p > -4
\]

\[
\Leftrightarrow p > -\frac{4}{5}
\]

Since \( p > 0 \), \( p > -\frac{4}{5} \) and hence the condition

\[
\int_c^\infty \left( \left( \frac{(-q)^{1/2}}{(-q)^{3/2}} \right)' \right) (-q)^{-1/4} dx < \infty
\]

is satisfied.

Now by Theorem 2.2, \( q(x) = kx^p \) is in the limit point case at \( \infty \) if and only if
\[
\int_{c}^{\infty} (-q(x))^{-1/2}dx = \infty.
\]

Now
\[
\int_{c}^{\infty} (-kx^p)^{-1/2} dx = (-k)^{-1/2} \int_{c}^{\infty} x^{-p/2} dx
\]
\[
= (-k)^{-1/2} \int_{c}^{\infty} \frac{1}{x^p} dx
\]

Hence
\[
\int_{c}^{\infty} (-kx^p)^{-1/2} dx = \infty
\]
\[
\iff p \leq 1
\]
\[
\iff p \leq 2.
\]

Therefore for \( p > 0 \) and \( k < 0 \), \( q(x) = kx^p \) is in the limit point case at \( \infty \) if and only if \( p \leq 2 \).

**Proposition 3.7**

For \( p > 2 \) and \( k < 0 \), \( \tau = -\frac{d^2}{dx^2} + kx^p \) is in the limit circle case at \( \infty \).

**Proof.** For \( p > 2 \), \( k < 0 \) and \( p > 0 \). Applying Theorem 3.6, we get \( \tau = -\frac{d^2}{dx^2} + kx^p \) is not in the limit point case at \( \infty \). *i.e.* \( \tau = -\frac{d^2}{dx^2} + kx^p \) is in the limit circle case at \( \infty \).

**Conclusion**

Following table illustrates how \( \tau \) lies in limit circle case or limit point case at \( \infty \) for \( q(x) = kx^p \).
### Classification of Sturm-Liouville problems at infinity

\[ p \quad k \quad \tau \text{ is in l.c.c./l.p.c.} \]

- \(2 \quad (-\infty, \infty) \quad \tau \text{ is in l.p.c.}\)
- \((-\infty, \infty) \quad 0 \quad \tau \text{ is in l.p.c.}\)
- \((-\infty, \infty) \quad (0, \infty) \quad \tau \text{ is in l.p.c.}\)
- \([0,2] \quad (-\infty, 0) \quad \tau \text{ is in l.p.c.}\)
- \((-\infty, 0) \quad (-\infty, 0) \quad \tau \text{ is in l.p.c.}\)
- \((2, \infty) \quad (-\infty, 0) \quad \tau \text{ is in l.c.c.}\)

### References


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