A Note on Subdifferential of Composed Convex Operator

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Abstract

The aim of this paper is to establish the strong vector subdifferential of the convex operator $f + g\circ h$ when $f$, $g$ and $h$ are vector valued convex mappings and $g$ is nondecreasing. An application to a cone-constrained vector optimization problem is also given.

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1 Introduction

It is well known that vector subdifferential calculus constitute one of the intensively developing directions in vector optimization. This area is recognized for its theoretical aspects as well as its wide range of applications. Recently, El Maghri et al. discussed in [1] the calculus rules of the efficient subdifferentials (Pareto and proper) for the sum and/or composition of convex vector valued mappings under some qualifications conditions by using the scalarization principle and the regular subdifferentiability. The question that we will attempt to develop in this note is to state the composition rule for strong vector subdifferential. This issue was not discussed by the authors in their work [1]. To our knowledge, it seems that this problem has not been explored previously except the scalar case (see for instance [2] and references therein) and an earlier contribution due to Théra [3] by establishing in the framework of ordered topological vector space, by using the so-called sandwich theorem, a formula for the strong vector subdifferential of the composed convex operator $g \circ h$ when $g$ and $h$ are convex and affine mappings respectively. The purpose of the present paper is to extend this formula by assuming $h$ a convex mapping and $g$ a convex nondecreasing mapping. The outline of the paper is as follows. The next section contains some preliminaries which are needed in the sequel. Section 3 contains the main results of this paper that provide a formula for the strong subdifferential $\partial^s(f + g \circ h)$ when $f$, $g$ and $h$ are vector valued convex mappings and $g$ is nondecreasing. In the last section, some necessary and sufficient conditions for the existence of strong minimizer of vector composed convex optimization problem are given.

2 Notations, definitions and preliminaries

Throughout this paper, $X, Y$ and $Z$ are Hausdorff locally convex spaces and $Y_+ \subset Y$ (resp. $Z_+ \subset Z$) be a nonempty convex cone introducing a partial order in $Y$ (resp. in $Z$) defined by

$$y_1 \leq_Y y_2 \iff y_2 - y_1 \in Y_+.$$  

We adjoin an abstract maximal element $+\infty$ to $Y$ (resp. to $Z$) such that

$$y \leq_Y +\infty, \quad \forall y \in Y.$$  

We extend in a natural way the addition and the scalar multiplication of $Y$ (resp. $Z$) to $Y \cup \{+\infty\}$ (resp. to $Z \cup \{+\infty\}$) by setting $y + (+\infty) = +\infty$ for all $y \in Y \cup \{+\infty\}$ and $\lambda (+\infty) = +\infty$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and also we adopt the convention $0 (+\infty) = +\infty$

For a given mapping $h : X \rightarrow Y \cup \{+\infty\}$ we denote by

$$\text{Epi } h := \{(x, y) \in X \times Y : h(x) \leq_Y y\}$$
its epigraph and by
\[ \text{dom } h := \{ x \in X : h(x) \in Y \} \]
its effective domain. When \( \text{dom } h \neq \emptyset \), we says that \( h \) is proper and the set
\[ \text{Im } h := h(\text{dom } h) = \{ h(x) : x \in \text{dom } h \} \]
is called the effective image set of \( h \). Since convexity plays an important role in the following investigations, let us recall the concept of cone-convex mappings. The mapping \( h : X \rightarrow Y \cup \{+\infty\} \) is said to be \( Y^+ \)-convex if for every \( \lambda \in [0,1] \) and \( x_1, x_2 \in X \)
\[ h(\lambda x_1 + (1 - \lambda)x_2) \leq_Y \lambda h(x_1) + (1 - \lambda)h(x_2). \]
A mapping \( g : Y \rightarrow Z \cup \{+\infty\} \) is said to be \((Y^+_+, Z^+_+)-\)nondecreasing, if for each \( y_1, y_2 \in Y \)
\[ y_1 \leq_Y y_2 \Rightarrow g(y_1) \leq_Z g(y_2). \]
The composed vector mapping \( g \circ h : X \rightarrow Z \cup \{+\infty\} \) is defined by
\[ (g \circ h)(x) := \begin{cases} g(h(x)) & \text{if } x \in \text{dom } h + \infty, \text{ otherwise} \\ +\infty, & \text{otherwise} \end{cases} \]
and its effective domain is therefore given by
\[ \text{dom } (g \circ h) = h^{-1}(\text{dom } g) \cap \text{dom } h. \]
It is easy to see that if \( g \) is \((Y^+_+, Z^+_+)-\)nondecreasing and \( Z^+_+\)-convex and \( h \) is \( Y^+\)-convex then \( g \circ h \) is \( Z^+_+\)-convex.

To relate the order structure and the topological structure of \( Z \), we demand furthermore that \( Z \) will be normal i.e. there exists a basis of neighborhoods \( V \) of the origin such that
\[ V = (V + Z^+_+)^\perp \cap (V - Z^+_+). \]
We need some further definitions. Let \( L(X,Y) \) stand for the real vector space of continuous linear mappings between \( X \) and \( Y \). Following [3], an element \( T \in L(X,Y) \) is called a strong subgradient of the \( Y^+\)-convex mapping \( h : X \rightarrow Y \cup \{+\infty\} \) at \( \bar{x} \in \text{dom } h \) if
\[ T(x - \bar{x}) \leq_Y h(x) - h(\bar{x}), \quad \forall x \in X. \]
The set of all strong subgradients of the mapping \( h \) at \( \bar{x} \in X \), denoted by \( \partial^s h(\bar{x}) \), is called the strong vector subdifferential whenever \( \bar{x} \in \text{dom } h \). We set \( \partial^s h(\bar{x}) = \emptyset \) whenever \( \bar{x} \notin \text{dom } h \). Let us note that when \( h \) is a convex function, \( \partial^s h(\bar{x}) \) reduces to the well known subdifferential
\[ \partial h(\bar{x}) := \{ x^* \in X^* : h(x) - h(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle, \forall x \in X \} \]
Let us point out that if \( h : X \rightarrow Y \cup \{+\infty\} \) is \( Y^+\)-convex and continuous at some point \( \bar{x} \in \text{dom } h \) then \( \partial^s h(\bar{x}) \neq \emptyset. \)
3 Subdifferential of composed convex operator

As mentioned in the introduction, Théra [3], established in the framework of ordered topological vector space by using the sandwich theorem, a formula for the strong subdifferential of \( g \circ h \) when \( g \) and \( h \) are convex and affine respectively. Our main goal in this section, is to extend this formula in the same setting by assuming \( h \) a convex mapping taking value in an ordered topological vector space.

In what follows, we will use the composed mapping \( T \circ h \) where \( T \in L(Y,Z) \) and \( h : X \rightarrow Y \cup \{+\infty\} \) is \( Y_+ \)-convex. For preserving the convexity of the composed mapping \( T \circ h \), we will need that the linear operator \( T \) satisfy the following condition

\[ T \in L_+(Y,Z) := \{T \in L(Y,Z) : T(Y_+) \subset Z_+\} \]

which ensures that \( T \) is \((Y_+,Z_+)\)-nondecreasing.

For this, let us establish the following lemma

**Lemma 3.1** Let \( g : Y \rightarrow Z \cup \{+\infty\} \) be \( Z_+ \)-convex and \((Y_+,Z_+)\)-nondecreasing mapping. Then we have

\[ \partial^s g(\bar{y}) \subset L_+(Y,Z) \]

for any \( \bar{y} \in Y \).

**Proof.** Suppose that \( \partial^s g(\bar{y}) \neq \emptyset \), then we have

\[ T \in \partial^s g(\bar{y}) \iff T(y - \bar{y}) \leq_Y g(y) - g(\bar{y}), \quad \forall y \in Y \]

(2.1)

By taking \( y \in Y_+ \) and substituting in (2.1) \( y \) by \( \bar{y} - y \), we obtain

\[ -T(y) \leq_Z g(\bar{y} - y) - g(\bar{y}), \quad \forall y \in Y \]

Since \( \bar{y} - y \leq_Y \bar{y} \) and that \( g \) is \((Y_+,Z_+)\)-nondecreasing, it follows that for any \( y \in Y_+ \), \( T(y) \in Z_+ \) i.e. \( T \in L_+(Y,Z) \). \( \square \)

The following corollary follows immediately from above lemma

**Corollary 3.2** Let \( h : X \rightarrow Y \cup \{+\infty\} \) be \( Y_+ \)-convex and \( g : Y \rightarrow Z \cup \{+\infty\} \) be \( Z_+ \)-convex and \((Y_+,Z_+)\)-nondecreasing. Then for any \( T \in \partial^s g(\bar{y}) \), the mapping \( T \circ h \) is \( Z_+ \)-convex.

The approach that we will use for computing the strong vector subdifferential of composed convex mapping is to transform it as the strong subdifferential of the sum of two vector convex mappings. In the literature, the calculus rule of the strong vector subdifferential of the sum of two convex vector mappings was established by Théra [3] and Zowe [4] in the setting of normal order complete vector topological space. Let us recall that a vector topological space \((Z,Z_+)\) is said to be order complete if for each subset \( A \subset Z \) order bounded from below, \( \inf A \) exists.
Theorem 3.3 \cite{3}, \cite{4} Let $X$ be a Hausdorff locally convex vector space and let $(Z, Z_+)$ be a normal order complete Hausdorff locally convex space. Suppose $g_1 : X \rightarrow Z \cup \{+\infty\}$ and $g_2 : X \rightarrow Z \cup \{+\infty\}$ are $Z_+$-convex mappings such that $g_1$ is finite and continuous at some point $a \in \text{dom} g_2$. Then, for any $x \in X$

$$\partial^s (g_1 + g_2)(x) = \partial^s g_1(x) + \partial^s g_2(x).$$

Let $f : X \rightarrow Z \cup \{+\infty\}$, $g : Y \rightarrow Z \cup \{+\infty\}$ and $h : X \rightarrow Y \cup \{+\infty\}$ be proper mappings. Let us consider the following auxiliary mappings

$$F : X \times Y \rightarrow Z \cup \{+\infty\} \quad (x, y) \rightarrow F(x, y) := f(x) + \delta_{\text{Epi} h}(x, y)$$

and

$$G : X \times Y \rightarrow Z \cup \{+\infty\} \quad (x, y) \rightarrow G(x, y) := g(y)$$

where $\delta_{\text{Epi} h} : X \times Y \rightarrow Z \cup \{+\infty\}$ stands for the indicator mapping defined for any $(x, y) \in X \times Y$ by

$$\delta_{\text{Epi} h}(x, y) := \begin{cases} 0 & \text{if } (x, y) \in \text{Epi} h + \infty, \\ +\infty, & \text{otherwise}. \end{cases}$$

Let us note that the vector indicator mapping appears to posses properties like the scalar one. When $g : Y \rightarrow Z \cup \{+\infty\}$ is $(Y_+, Z_+)$-nondecreasing, one has for any $(x, y) \in X \times Y$

$$(f + g \circ h)(x) \leq Z f(x) + g(y) + \delta_{\text{Epi} h}(x, y),$$

and since the vector space $(Z, Z_+)$ is order complete, it follows that for any $x \in X$

$$(f + g \circ h)(x) = \inf_{y \in Y} \{f(x) + g(y) + \delta_{\text{Epi} h}(x, y)\} = \inf_{y \in Y} \{F(x, y) + G(x, y)\}.$$
Theorem 3.4 Let $f : X \rightarrow Z \cup \{+\infty\}$ be proper $Z_+$-convex, $g : Y \rightarrow Z \cup \{+\infty\}$ be proper $Z_+$-convex and $(Y_+, Z_+)$-nondecreasing and $h : X \rightarrow Y \cup \{+\infty\}$ be proper $Y_+$-convex. If there exists some point $a \in \text{dom } f \cap \text{dom } h$ such that $g$ is finite and continuous at point $h(a) \in Z$ then

$$
\partial^s(f + g \circ h)(\bar{x}) = \bigcup_{T \in \partial^s(h(\bar{x}))} \partial^s(f + T \circ h)(\bar{x}), \tag{2.3}
$$

for any $\bar{x} \in X$.

Proof. At first we show that the following assertions i) and ii) are equivalents

i) $(T, -K) \in \partial^s F(\bar{x}, \bar{y})$ and $(0, K) \in \partial^s G(\bar{x}, \bar{y})$

ii) $K \in \partial^s g(\bar{y})$ and $T \in \partial^s (f + K \circ h)(\bar{x})$.

Indeed, suppose that i) holds. It is easy to see that $K \in \partial^s g(\bar{y}) \iff (0, K) \in \partial^s G(\bar{x}, \bar{y})$.

As $(T, -K) \in \partial^s F(\bar{x}, \bar{y})$ we have for any $x \in X$ and $y \in Y$

$$
T(\bar{x} - x) - K(y - \bar{y}) \leq_Z f(x) + \delta_{\text{Epi} h}(x, y) - f(\bar{x}) - \delta_{\text{Epi} h}(\bar{x}, \bar{y}). \tag{2.4}
$$

By taking for any $y \in Y_+$, $x = \bar{x}$ and $\bar{y} + y$ in place of $y$, we get $Ky \in Z_+$ for any $y \in Y_+$ i.e. $K \in L_+(Y, Z)$. So, $K \circ h$ is $Z_+$-convex. Taking now $y := h(x)$ for any $x \in \text{dom } h$ in (2.3), we have

$$
T(\bar{x} - x) \leq_Z f(x) + (K \circ h)(x) - f(\bar{x}) - (K \circ h)(\bar{x})
$$

and hence $T \in \partial^s (f + K \circ h)(\bar{x})$. Conversely, let $T \in \partial^s (f + K \circ h)(\bar{x})$, then for any $x \in \text{dom } h$, one has

$$
T(\bar{x} - x) \leq_Z f(x) + (K \circ h)(x) - f(\bar{x}) - (K \circ h)(\bar{x})
$$

and since $K \in \partial^s g(\bar{y}) \subset L_+(Y, Z)$, we obtain for any $(x, y) \in \text{Epi} h$

$$
T(\bar{x} - x) \leq_Z f(x) + K(y) - f(\bar{x}) - K(\bar{y}).
$$

Therefore for any $(x, y) \in X \times Y$ one has

$$
T(\bar{x} - x) - K(y - \bar{y}) \leq_Z f(x) + \delta_{\text{Epi } h}(x, y) - f(\bar{x}) - \delta_{\text{Epi } h}(\bar{x}, \bar{y})
$$

and then

$$(T, -K) \in \partial^s F(\bar{x}, \bar{y}).$$

Now, we have to show that the equality (2.3) holds for any $\bar{x} \in X$. It is easy to check that the following inclusion

$$
\bigcup_{T \in \partial^s g(h(\bar{x}))} \partial^s(f + T \circ h)(\bar{x}) \subset \partial^s(f + g \circ h)(\bar{x})
$$
is satisfied without constraint qualification. Conversely, by taking \( \bar{x} \in \text{dom } h \) and \( \bar{y} := h(\bar{x}) \), one has from (2.2)
\[
T \in \partial^s(f + g \circ h)(\bar{x}) \iff (T,0) \in \partial^s(F + G)(\bar{x},\bar{y}).
\]

It is obvious to see that if there exists some point \( a \in \text{dom } f \cap \text{dom } h \) such that \( g \) is finite and continuous at point \( h(a) \in Z \) then \( G \) is finite and continuous at point \( (a,h(a)) \in \text{dom } F \) and hence according to Theorem 3.1, we have
\[
\partial^s(F + G)(\bar{x},\bar{y}) = \partial^s F(\bar{x},\bar{y}) + \partial^s G(\bar{x},\bar{y}).
\]

From the above equivalence of assertions i) and ii), it follows that \( T \in \partial^s(f + g \circ h)(\bar{x}) \)
if and only if there exist \((T_1,-K) \in \partial^s F(\bar{x},\bar{y})\) and \((T_2,M) \in \partial^s G(\bar{x},\bar{y})\) such that \( (T,0) = (T_1,-K) + (T_2,M) \). The definition of \( G \) ensures that \( T_2 = 0 \) which yields \( K \in \partial^s g(\bar{y}) \) and \( T \in \partial^s(f + K \circ h)(\bar{x}) \). Thanks to Lemma 3.2, we obtain
\[
T \in \partial^s(f + K \circ h)(\bar{x}) \quad \text{and} \quad K \in \partial^s g(\bar{y}).
\]

So
\[
\partial^s(f + g \circ h)(\bar{x}) = \bigcup_{T \in \partial^s g(\bar{y})} \partial^s(f + T \circ h)(\bar{x})
\]

which completes the proof.

In particular by taking \( f \equiv 0 \), we have

**Corollary 3.5** Let \( g : Y \to Z \cup \{+\infty\} \) be proper \( Z_+ \)-convex and \( (Y_+,Z_+) \)-nondecreasing and \( h : X \to Y \cup \{+\infty\} \) be proper \( Y_+ \)-convex. If there exists some point \( a \in \text{dom } h \) such that \( g \) is finite and continuous at point \( h(a) \in Z \) then
\[
\partial^s(g \circ h)(\bar{x}) = \bigcup_{T \in \partial^s g(\bar{y})} \partial^s(T \circ h)(\bar{x})
\]

for any \( \bar{x} \in X \).

Consider now the case of composition with an affine operator \( h : X \to Y \) associated to a linear operator \( A : X \to Y \) and let \( g : Y \to Z \cup \{+\infty\} \) be a proper and \( Z_+ \)-convex mapping. Put \( Y_+ := \{0_Y\} \), obviously the mapping \( g \) is \( (Y_+,Z_+) \)-nondecreasing on \( Y \). So, applying Corollary 3.2, one gets the following result

**Corollary 3.6** Let \( g : Y \to Z \cup \{+\infty\} \) be a proper and \( Z_+ \)-convex mapping and \( h : X \to Y \) be an affine continuous operator associated to a linear operator \( A \in L(X,Y) \). Assume that there exists some \( a \in \text{dom } h \) such that \( g \) is finite and continuous at \( h(a) \). Then for every \( \bar{x} \in X \), one has
\[
\partial^s(g \circ h)(\bar{x}) = A^*(\partial^s g(h(\bar{x}))),
\]
where \( A^* : Y^* \to X^* \) is the adjoint operator of \( A \).
Suppose now the case of composition with a linear operator \( A : X \rightarrow Y \) with domain \( D_A \) (a vector subspace of \( X \)). By setting
\[
h(x) := \begin{cases} 
Ax & \text{if } x \in D_A \\
+\infty & \text{otherwise}
\end{cases}
\]
and \((g \circ A)(x) = +\infty \) if \( x \notin D_A \), then \( g \circ A = g \circ h \). So, applying Corollary 3.2, one gets the following result

**Corollary 3.7** Let \( g : Y \rightarrow Z \cup \{+\infty\} \) be a proper and \( Z_+ \)-convex mapping and \( A : X \rightarrow Y \) be a linear operator with domain \( D_A \). Assume that \( g \) is finite and continuous at some point of \( \text{Im} \ A \). Then for every \( \bar{x} \in D_A \) and with the above definition of \( g \circ A \) over all the space \( X \), one has
\[
\partial^s(g \circ A)(\bar{x}) = \{ T \in L(X,Z) : \exists K \in \partial^s g(A\bar{x});\ T|_A = K \circ A \}.
\]
Here \( T|_A \) denotes the restriction of \( T \) to \( D_A \).

In the case \( D_A \) is dense in \( X \) we obtain the following corollary

**Corollary 3.8** Let \( g : Y \rightarrow Z \cup \{+\infty\} \) be a proper and \( Z_+ \)-convex mapping and \( A : X \rightarrow Y \) be a densely defined linear operator. Assume that \( g \) is finite and continuous at some point of \( \text{Im} \ A \). Then one has for every \( \bar{x} \in D_A \)
\[
\partial^s(g \circ A)(\bar{x}) = A^*(\partial^s g(A\bar{x}) \cap D_{A^*}).
\]

4 Application to vector optimization problem

In this section, we will consider vector optimization problem of the form
\[
(P) \quad \inf_{x \in X} (f + g \circ h)(x)
\]
where \( f : X \rightarrow Z \cup \{+\infty\} \) is proper \( Z_+ \)-convex, \( g : Y \rightarrow Z \cup \{+\infty\} \) is proper \( Z_+ \)-convex and \((Y_+,Z_+)\)-nondecreasing and \( h : X \rightarrow Y \cup \{+\infty\} \) is proper \( Y_+ \)-convex.

We will apply the preceding results to obtain optimality conditions of vector optimization problem \((P)\). A point \( \bar{x} \) is said to be strong minimizer of problem \((P)\) if
\[
(f + g \circ h)(\bar{x}) \leq_Z (f + g \circ h)(x), \quad \forall x \in X
\]

**Proposition 4.1** If there exists some point \( a \in \text{dom} \ f \cap \text{dom} \ h \) such that \( g \) is finite and continuous at \( h(a) \in Z \), then \( \bar{x} \) is a strong minimizer of the problem \((P)\) if and only if there exists some \( T \in \partial^s g(h(\bar{x})) \) such that \( 0 \in \partial^s(f + T \circ h)(\bar{x}) \).
Proof. We have $\bar{x}$ is a strong minimizer of the problem (P) if and only if $0 \in \partial^s(f + g \circ h)(\bar{x})$. Thanks to Theorem 3.2, there exists some $T \in \partial^s g(h(\bar{x}))$ such that $0 \in \partial^s(f + T \circ h)(\bar{x})$. □

Now, let us consider the cone-constrained vector optimization problem

$$(Q) \left\{ \begin{array}{l}
\inf f(x) \\
h(x) \in -Y_+
\end{array} \right.$$ 

where $f : X \rightarrow Z \cup \{+\infty\}$ is proper $Z_+$-convex and $h : X \rightarrow Y \cup \{+\infty\}$ is proper $Y_+$-convex. By introducing the vector indicator mapping $\delta_{-Y_+} : Y \rightarrow Z \cup \{+\infty\}$, the problem (Q) may be rewritten equivalently as the unconstrained composed convex problem (P) by setting

$$g : Y \rightarrow Z \cup \{+\infty\},$$

$$y \rightarrow g(y) := \delta_{-Y_+}(y).$$

By considering the normal cone of $-Y_+$ (at $\bar{y} \in -Y_+$) defined by

$$N_{-Y_+}^s(\bar{y}) := \partial^s \delta_{-Y_+}(\bar{y}) = \{T \in L(Y, Z) : T(y - \bar{y}) \leq_2 0, \ ∀y \in -Y_+\}$$

and if we supose in the sequel that the cone $Z_+$ is pointed i.e. $Z_+ \cap -Z_+ = \{0\}$, we have

**Lemma 4.2** i) The indicator mapping $\delta_{-Y_+} : Y \rightarrow Z \cup \{+\infty\}$ is $Z_+$-convex, proper and $(Y_+, Z_+)$-nondecreasing.

ii) $N_{-Y_+}^s(\bar{y}) = \{T \in L_+(Y, Z) : T(\bar{y}) = 0\}$.

**Proof.** i) The convexity and properness of $\delta_{-Y_+}$ are obvious since $Y_+$ is convex and $\text{dom} \delta_{-Y_+} = -Y_+ \neq \emptyset$. For the monotonicity of $\delta_{-Y_+}$, let us take any $y_1, y_2 \in Y$ such that $y_1 \leq_1 y_2$. If $y_2 \notin -Y_+$, obviously $\delta_{-Y_+}(y_1) \leq_2 \delta_{-Y_+}(y_2) = +\infty$. The case $y_2 \in -Y_+$ entails $y_1 \in -Y_+$ since $y_1 = y_1 - y_2 + y_2 \in -Y_+ - Y_+ \subset -Y_+$ and hence we get $\delta_{-Y_+}(y_1) = \delta_{-Y_+}(y_2) = 0$.

ii) We have

$$T \in N_{-Y_+}^s(\bar{y}) \iff T(y - \bar{y}) \leq_2 0, \ ∀y \in -Y_+.$$ 

By taking successively $y = 0$ and $y = 2\bar{y}$ we obtain respectively $T(\bar{y}) \in Z_+$ and $T(\bar{y}) \in -Z_+$ and since the cone $Z_+$ is pointed we get $T(\bar{y}) = 0$. Consequently, $T(y) \geq_2 0$, for any $y \in Y_+$ i.e. $T \in L_+(Y, Z)$. The reverse inclusion is immediate since $T(y - \bar{y}) = T(y) \leq_2 0$, for any $y \in -Y_+$. □

Now, we are ready to state necessary and sufficient optimality conditions associated to vector problem (Q)
Proposition 4.3 If there exists some $a \in \text{dom } f \cap \text{dom } h$ such that $h(a) \in -\text{int} Y_+$ then $\bar{x}$ is a strong minimizer of the problem (Q) if and only if there exists some $T \in L_+(Y, Z)$ satisfying

\[
\begin{align*}
  h(\bar{x}) &\in -Z_+ \\
  T(h(\bar{x})) &= 0 \\
  0 &\in \partial^s(f + T \circ h)(\bar{x}).
\end{align*}
\]

(Here $\text{int} Y_+$ stands for the topological interior of $Y_+$).

Proof. $\bar{x}$ is a strong minimizer of (Q) if and only if $0 \in \partial^s(f + \delta_{-Y_+} \circ h)(\bar{x})$ and by virtue of Theorem 3.2 and Lemma 4.1 there exists $T \in L_+(Y, Z)$ such that $h(\bar{x}) \in -Z_+$, $T(h(\bar{x})) = 0$ and $0 \in \partial^s(f + T \circ h)(\bar{x})$. \qed

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