Regulated Function on a Cell
in the $n$-Dimensional Space to Hilbert Space

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Abstract

In this paper we generalize some characteristics of regulated function from a
cell in the $n$-dimensional Euclidean space $\mathbb{R}^n$ to Hilbert space. We will give some
sufficient conditions of a regulated function to have its integral. Some properties of
the integral and some convergence theorems will be stated. Furthermore, by defining
characteristics function and corresponding interval function of its point function,
we prove the integrability of step function $f$ with respect to a step function $g$.
The value of the integral can be determined explicitly by the inner product on $X$.

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and step function

1. INTRODUCTION

A real valued function on $[a, b]$ is said to be regulated if and only if $f(x^-)$ exists
for every $x \in (a, b]$ and $f(x^+)$ exists for every $x \in [a, b)$ [2]. Regulated function
has important role in differential equation problems with singularity [10]. In
1996, Tvrdý used regulated function to have a bounded linear functional on
regular regulated function space. He used Perron integral [11].

Lovelady in 1971 gave sufficient and necessary conditions of the existence
of solution for non-homogeneous linear integral equations by using Lebesgue
integral [9]. The Lebesgue integral is equivalent with McShane integral. In
the theory of integral, every Lebesgue (McShane) integrable function on \([a, b]\) is
Henstock-Kurzweil integrable on \([a, b]\) [8].

In 2009 and 2012, Indrati generalized the definition of real regulated function
on \([a, b]\) to be real regulated function on a cell \(E\) in the \(n\)-dimensional Euclidean
space [4, 5]. In the \(n\)-dimensional Euclidean space, we do not have left and
right limit. The generalization is not straightforward. It is defined by the
characteristic of the regulated function that every regulated function can be
uniformly approximated by step function.

Brokate and Krejčí generalized a real regulated function to be Hilbert regu-
lated function. They gave a Riemann-Stieltjes type of Young-Stieltjes integral
from \([a,b]\) to a Hilbert space \(X\). They focus on \(BV[a,b]\) [1]. Krejčí used
Henstock-Stieltjes for a function from \([a,b]\) to a Hilbert Space in the discus-
sion of An application of non-smooth mechanics in real analysis [7]. In [6],
Indrati give the Henstock-Stieltjes type a Young-Stieltjes integral by using
\(BV_p[a,b]\).

The results of the researches in regulated function give an opportunity to
work on a function from a cell in \(n\)-dimensional Euclidean space to Hilbert
space. In this paper, we generalize the definition and some characteristics of
regulated function from a cell in the \(n\)-dimensional Euclidean space to Hilbert
space. The results will be used to generalize the definition and some properties
of the integral from a cell in the \(n\)-dimensional Euclidean space to Hilbert
space due to regulated function to investigate the integrability of regulated
function. Furthermore, some convergence theorems will be stated. The result
of this research will solve some problems that involve singularity, such as in
differential problems, optimization, representation theorems, etc.

2. Preliminary Notes

Before discussing the results of the research, we restate some concepts that
will be used in the generalization.

Let \(a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n\). We have four types of
interval in \(\mathbb{R}^n\).

(i) \([a, b] = \{x = (x_1, x_2, \ldots, x_n) : a_i \leq x_i \leq b_i, i = 1, 2, 3, \ldots, n\}\).

(ii) \((a, b] = \{x = (x_1, x_2, \ldots, x_n) : a_i < x_i \leq b_i, i = 1, 2, 3, \ldots, n\}\).

(iii) \([a, b) = \{x = (x_1, x_2, \ldots, x_n) : a_i \leq x_i < b_i, i = 1, 2, 3, \ldots, n\}\).

(iv) \((a, b) = \{x = (x_1, x_2, \ldots, x_n) : a_i < x_i < b_i, i = 1, 2, 3, \ldots, n\}\).

In this discussion, a cell \(E\) stands for a non-degenerate closed and bounded
interval \([a, b]\) in the Euclidean space \(\mathbb{R}^n\), where \(a < b\). Its volume will be
represented by \(|E|\).

For \(x = (x_1, x_2, \ldots, x_n) \in E\), \(\|x\| = \max |x_k| : 1 \leq k \leq n\). If \(E\) is a cell and
\(\delta\) is a positive function on \(E\), an open ball centered at \(x \in E\) with radius \(\delta(x)\),
in short will be written \(B(x, \delta(x))\), is defined as follows:

\[
B(x, \delta(x)) = \{y : \|x - y\| < \delta(x)\}.
\]
A collection of cells \( \{ I_i : i = 1, 2, \ldots, p \} \) is called non-overlapping if \( I_i \cap I_j = \emptyset \) for \( i \neq j \). A collection of finite non-overlapping cells, \( \mathcal{D} = \{ I \} \) with \( \cup_{I \in \mathcal{D}} I = E \), is called a partition of \( E \). A collection \( \mathcal{D} = \{ (I, x) \} = \{ (I_1, x_1), (I_2, x_2), \ldots, (I_p, x_p) \} \) is called a \( \delta \)-fine partition of \( E \) if \( E = \cup_{I \in \mathcal{D}} I \), \( x_i \in I_i \subseteq B(\delta(x_i)) \), and \( I_i \cap I_j = \emptyset \), \( i \neq j \), \( i = 1, 2, \ldots, p \). Furthermore, \( (I, x) \in \mathcal{D} \) is called a \( \delta \)-fine interval with associated point \( x \). A collection \( \mathcal{D} = \{ (I, x) \} = \{ (I_1, x_1), (I_2, x_2), \ldots, (I_p, x_p) \} \) is called a \( \delta \)-fine partial partition in \( E \) if \( \cup_{I \in \mathcal{D}} I \subseteq E \), \( x_i \in I_i \subseteq B(\delta(x_i)) \), and \( I_i \cap I_j = \emptyset \), \( i \neq j \), \( i = 1, 2, \ldots, p \).

3. Main Results

The discussion is started by generalizing the concept of the regulated function from real-valued function to be Hilbert-valued function.

3.1. Regulated function in the \( n \)-dimensional space. The generalization of the definition of regulated function is started by defining step function from a cell \( E \) to Hilbert space \( X \). Some properties of regulated function will be done by considering facts that every cell is compact and a continuous function from a compact set is uniformly continuous.

**Definition 3.1.** A bounded function \( \varphi \) from a cell \( E \subseteq \mathbb{R}^n \) to a Hilbert \( X \) is said to be a step function on \( E \), if it assumes only a finite number of distinct values in \( X \), where the non-zero value being taken on an interval.

If \( \varphi \) and \( \psi \) are step functions from a cell \( E \subseteq \mathbb{R}^n \) to a Hilbert \( X \), then \( \alpha \varphi \) and \( \varphi + \psi \) are step functions from a cell \( E \subseteq \mathbb{R}^n \) to \( X \), for every scalar \( \alpha \).

**Lemma 3.2.** Let \( E \subseteq \mathbb{R}^n \) be a cell and \( X \) be a Hilbert space. A bounded function \( \varphi : E \rightarrow X \) is a step function on a cell \( E \) if and only if there exist a partition \( \mathcal{D} = \{ I \} = \{ I_1, I_2, \ldots, I_p \} \) and \( c_i \in \mathbb{R} \), \( i = 1, 2, 3, \ldots, p \), such that \( f(x) = c_i \), \( x \in I_i \).

In this writing, in Hilbert \( X \), for every \( x \in X \),

\[ \|x\|_X = \sqrt{\langle x, x \rangle} . \]

**Definition 3.3.** Let \( E \subseteq \mathbb{R}^n \) be a cell and \( X \) be a Hilbert space. A function \( f : E \rightarrow X \) is called a regulated function on \( E \), if for every \( \epsilon > 0 \), there exists a step function \( \varphi : E \rightarrow X \) such that for every \( x \in E \) we have

\[ \|f(x) - \varphi(x)\|_X < \epsilon . \]

It is clear that every step function from a cell \( E \) to a Hilbert space \( X \) is a regulated function on \( E \). In Theorem 3.4, we prove that a continuous function is regulated.

**Theorem 3.4.** If \( f \) is a continuous function on a cell \( E \subseteq \mathbb{R}^n \), then \( f \) is a regulated function on \( E \).
Proof. Let $\epsilon > 0$ be given. Since $f$ is continuous on $E$, we have $f$ is uniformly continuous on $E$. There is a positive constant $\delta$ such that for every $x, y \in E$, $\|x - y\| < \delta$, we have

$$\|f(x) - f(y)\|_X < \epsilon.$$ 

Let $P = \{I_1, I_2, \ldots, I_p\}$, where $I_i = [a_i, b_i], i = 1, 2, 3, \ldots, p$, be a partition of $E$, where $\|P\| < \delta$. If $x \in E$, then there is $i, 1 \leq i \leq p$ such that $x \in I_i^o$ or there exist some $i, 1 \leq i \leq p$ such that $x \in \partial(I_i)$. Let $J_x = \{i : 1 \leq i \leq p, x \in \partial(I_i)\}$. Furthermore, we define a function $\varphi : E \to X$, with

$$\varphi(x) = \begin{cases} f(a_i), & \text{for } x \in I_i^o \text{ for some } i, 1 \leq i \leq p \\ f(a_s), & \text{for } x \in \partial(I_s), \end{cases}$$

where $s = \sup\{i : i \in J_x\}$. By the definition of $\varphi$, for every $x \in E$, we have

$$\|\varphi(x) - f(x)\|_X < \epsilon. \square$$

The characterization of the regulated function has been done based on the Definition 3.3.

From the definition of regulated function, we have a characteristic of regulated function in a sequence of step functions in Theorem 3.5.

Start from now, $E$ and $X$ denote a cell in $\mathbb{R}^n$ and a Hilbert space, respectively.

**Theorem 3.5.** A function $f : E \to X$ is regulated if and only if there exists a sequence of step functions $\{\varphi_k\}$ that converges uniformly to $f$ on $E$.

**Theorem 3.6.** A regulated function $f : E \to X$ is bounded.

**Lemma 3.7.** If $\varphi, \phi : E \to X$ are step functions, then $\varphi + \phi$ and $\alpha \varphi$ are step functions.

As corollary we have Theorem 3.8.

**Theorem 3.8.** If $f_1, f_2 : E \to X$ are regulated functions, then $f_1 + f_2$ and $\alpha f_1$ are regulated on $E$ for every $\alpha \in \mathbb{R}$.

Let $RF(E, X)$ stand for a collection of all regulated functions on a cell $E \subseteq \mathbb{R}^n$ to a Hilbert space $X$. From the definition of a regulated function from a cell $E$ to a Hilbert space $X$, we have $RF(E, X)$ is the closure of the set of all step functions on $E$. Furthermore, by Lemma 3.7, $RF(E, X)$ is a closed convex hull of a set of all linear combination of step functions on $E$. Therefore, $RF(E, X)$ is closed.

Theorem 3.8 implies that $RF(E, X)$ is a linear space over $\mathbb{R}$. From Theorem 3.2, it is clear that $C(E, X)$ is a subspace of $RF(E, X)$, where $C(E, X)$ is a collection of all continuous functions from a cell $E$ to a Hilbert space $X$.

The next characteristic of a regulated function will be used to prove the integrability of a regulated function in the sense of Henstock-Stieltjes.
Theorem 3.9. A function $f: E \to X$ is regulated if and only if for every $\epsilon > 0$, there exists a partition $\mathcal{D} = \{I\} = \{I_1, I_2, \ldots, I_p\}$ of $E$ such that for every $x, y \in I_i^o$, $1 \leq i \leq p$, we have

$$\|f(x) - f(y)\|_X < \epsilon.$$ 

Proof. ($\Rightarrow$) Let $\epsilon > 0$ be given. There exist a step function $\varphi$ on $E$, such that for every $x \in E$, we have

$$\|\varphi(x) - f(x)\|_X < \frac{\epsilon}{2}.$$ 

Let $\mathcal{D} = \{I\} = \{I_1, I_2, \ldots, I_p\}$ be the partition of $E$ due to the step function $\varphi$ on $E$. Therefore, for any $x, y \in I_i^o$, we have

$$|f(x) - f(y)| \leq |f(x) - \varphi(x)| + |\varphi(y) - f(y)| < \epsilon.$$ 

($\Leftarrow$) Let $\epsilon > 0$ be given. From the hypothesis, there is a partition $\mathcal{D} = \{I\} = \{I_i = [a_i, b_i], i = 1, 2, 3, \ldots, p\}$ of $E$ such that for every $x, y \in I_i^o$, $1 \leq i \leq p$, we have

$$\|f(x) - f(y)\|_X < \epsilon.$$ 

We define a function $\varphi$ on $E$, with

$$\varphi(x) = \begin{cases} f(a_i), & \text{for } x \in I_i^o \\ f(a_s): & x \in \partial(I_s), \end{cases}$$

where $s = \sup\{i : x \in \partial(I_i)\}$. We have $\varphi$ is a step function on $E$ and for every $x \in E$,

$$\|\varphi(x) - f(x)\|_X < \epsilon.$$

\[ \square \]

3.2. The Henstock-Stieltjes integral. In this section we generalize the Henstock-Stieltjes integral of a real-valued function on a cell $E$ to a function from a cell $E$ to a Hilbert space $X$. We do not generalize in the Bochner integral, here we define the integral by using the inner product of $X$. We investigate the integrability of regulated function from $E$ to $X$ with respect to a bounded variation function from $E$ to $X$. Moreover, some convergence theorems involving regulated functions are given in the last part of this section.

Some properties of the Henstock-Stieltjes integral will be given after the generalization its definition. We generalize the concept of a function has bounded variation as in Definition 3.11.

Definition 3.10. [?] Let $E \subseteq \mathbb{R}^n$ be a cell, $X$ be a Hilbert space, and $\mathcal{I}(E)$ be a collection of all intervals subset of $E$. A function $g: \mathcal{I}(E) \to X$ is called additive on $E$, if for every $I, J \in \mathcal{I}(E)$, $I^o \cap J^o = \emptyset$, and $I \cup J \in \mathcal{I}(E)$, we have

$$g(I \cup J) = g(I) + g(J).$$

Definition 3.11. Let $E \subseteq \mathbb{R}^n$ be a cell, $X$ be a Hilbert space, and $\mathcal{I}(E)$ be a collection of all intervals subset of $E$ and $g: \mathcal{I}(E) \to \mathbb{R}$ be an additive
function on $E$. The variation of $g$ on $E$, written $V_g(E)$ or $V(g, E)$, is defined as the following

$$V_g(E, X) = \sup_{D \in \mathcal{D}} \sum \|g(D)\|_X,$$

where the supremum is taken over all partitions $\mathcal{D} = \{D\}$ of $E$. We have $g$ has bounded variation on $E$, if $V_g(E, X) < \infty$, i.e., there is a constant $M \geq 0$ such that for every partition $\mathcal{D} = \{I\}$ of $E$, we have

$$\langle \mathcal{D} \rangle \sum \|g(I)\|_X \leq M.$$

In this section, we will consider that the function $g : E \rightarrow X$ is additive on a cell $E$.

In the Riemann-Stieltjes in the real line, there is no guarantee of the integrability of a function with respect to a function when they have a point of discontinuity. The Henstock-Stieltjes in the real line gives a guarantee of that case [3]. In this paper we give a generalized result in Theorem 3.15.

**Definition 3.12.** Let $E \subseteq \mathbb{R}^n$ be a cell and $X$ be a Hilbert space. A function $f : E \rightarrow X$ is said to be Henstock-Stieltjes integrable with respect to a function $g : \mathcal{J}(E) \rightarrow X$ on $E$, if there exists a real number $A$, such that for every $\epsilon > 0$, there is a positive function $\delta$ on $E$, such that for every $\delta$-fine partition $\mathcal{D} = \{(I, x)\} = \{(I_1, x_1), (I_2, x_2), \ldots, (I_p, x_p)\}$ of $E$, we have

$$|\langle \mathcal{D} \rangle \sum \langle f(x), g(I) \rangle - A| < \epsilon,$$

where $\langle \mathcal{D} \rangle \sum \langle f(x), g(I) \rangle = \sum_{i=1}^{p} \langle f(x_i), g(I_i) \rangle$.

The real number $A$ in Definition 3.12 is unique and is called the Henstock-Stieltjes integral value of $f$ with respect to $g$ on $E$, written

$$A = (HS) \int_{E} \langle f(x), dg(I) \rangle = (HS) \int_{E} \langle f, dg \rangle.$$

Here, $HS(E, X; g)$ represents a collection of all Henstock-Stieltjes integrable functions from a cell $E$ to a Hilbert space $X$ with respect to a function $g$ on a cell $E \subseteq \mathbb{R}^n$. Some basic properties of the Henstock-Stieltjes integral in [5] and [6] could be generalized successfully.

(i) $HS(E, X; g)$ is a linear space over $\mathbb{R}$.

(ii) If $f \in HS(E, X; g)$ and $f \in HS(E, X; h)$, then $f \in HS(E, X; g + h)$. Moreover, we have

$$(HS) \int_{E} \langle f, d(g + h) \rangle = (HS) \int_{E} \langle f, dg \rangle + (HS) \int_{E} \langle f, dh \rangle.$$

(iii) Let $E_1$ and $E_2$ be non-overlapping cells in $\mathbb{R}^n$ with $E_1 \cup E_2$ is a cell $E$. If $f \in HS(E_1, X; g)$ and $f \in HS(E_2, X; g)$, then $f \in HS(E, X; g)$. Moreover, we have

$$(HS) \int_{E} \langle f, dg \rangle = (HS) \int_{E_1} \langle f, dg \rangle + (HS) \int_{E_2} \langle f, dg \rangle.$$
Sometimes, it is easier to use the Cauchy’s Criterion to check the integrability of a function rather than to find the integral value of the function. The Cauchy’s Criterion still hold as stated in Theorem 3.13. The proof of the theorem is similar with the proof in real-valued function.

**Theorem 3.13. Cauchy’s Criterion**

A function \( f \) is Henstock-Stieltjes integrable with respect to \( g \) on a cell \( E \subseteq \mathbb{R}^n \) if and only if for every \( \epsilon > 0 \) there exists a positive function \( \delta \) on \( E \) such that for every two \( \delta \)-fine partitions \( \mathcal{D}_1 = \{(I_i, x_i)\} \) and \( \mathcal{D}_2 = \{(I_i, x_i)\} \) of \( E \), we have

\[
\left| (\mathcal{D}_1) \sum_{x} \langle f(x), g(I) \rangle - (\mathcal{D}_2) \sum_{x} \langle f(x), g(I) \rangle \right| < \epsilon.
\]

By Cauchy’s Criterion, we have Theorem 3.14.

**Theorem 3.14.** If \( f \in HS(E, X; g) \), then \( f \in HS(I, X; g) \) for every cell \( I \subseteq E \).

We prove the integrability of regulated function. with respect to a bounded variation function in Theorem 3.15.

**Theorem 3.15.** Let \( E \subseteq \mathbb{R}^n \) be a cell and \( X \) be a Hilbert space. If \( f : E \to X \) is a regulated function and \( g : \mathcal{J}(E) \to X \) is a bounded variation function on \( E \), then \( f \) is Henstock-Stieltjes integrable with respect to \( g \) on \( E \).

**Proof.** Let \( \epsilon > 0 \) be given. Since \( f \) is regulated on \( E \), by Theorem 3.9, there exists a partition \( \mathcal{D} = \{I\} = \{I_1, I_2, \ldots, I_p\} \) of \( E \) such that for every \( x, y \in I_i \), \( 1 \leq i \leq p \), we have

\[
\|f(x) - f(y)\| < \frac{\epsilon}{M},
\]

where \( M = V_g(E, X) \). We define a positive function \( \delta \) on \( E \) such that for every \( x \in I_i \), \( B(x, \delta(x)) \subseteq I_i \), for \( i = 1, 2, 3, \ldots, p \). Therefore, for any two \( \delta \)-fine partition \( \mathcal{D}_1 = \{(I_i, x_i)\} = \{(I_1, x_1), (I_2, x_2), \ldots, (I_p, x_p)\} \) and \( \mathcal{D}_2 = \{(J, y)\} = \{(J_1, y_1), (J_2, y_2), \ldots, (J_m, y_m)\} \) of \( E \), we have

\[
|(\mathcal{D}_1) \sum_{x} \langle f(x), g(I) \rangle - (\mathcal{D}_2) \sum_{x} \langle f(x), g(J) \rangle| \leq \sum_{i=1}^{p} \sum_{j=1}^{m} \|f(x_i) - f(y_j)\| \|g(I_i \cap J_j)\| \leq \frac{\epsilon}{M} \sum_{i=1}^{p} \sum_{j=1}^{m} \|g(I_i \cap J_j)\| \leq \frac{\epsilon}{M} \sum_{i=1}^{p+m} \|g(D_i)\| = \epsilon.
\]

By Cauchy’s Criterion, \( f \) is Henstock-Stieltjes integrable with respect to \( g \) on \( E \).
Some convergence theorems of the integral are given in the sense of regulated function.

**Theorem 3.16.** Let \( \{f_k\} \) be a sequence of regulated functions from a cell \( E \subseteq \mathbb{R}^n \) to a Hilbert space \( X \). If \( \{f_k\} \) uniformly converges to a function \( f : E \to X \) on \( E \), then \( f \) is regulated on \( E \).

**Proof.** Let \( \epsilon > 0 \) be given. For every \( k \), there is a step function \( \varphi_k : E \to \mathbb{R} \) such that for every \( x \in E \) we have
\[
\|f_k(x) - \varphi_k(x)\|_X < \epsilon.
\]
Since \( \{f_k\} \) uniformly converges to \( f \) on \( E \), there exists a positive integer \( k_0 \) such that for any \( k \in \mathbb{N} \), \( k \geq k_0 \), we have
\[
\|f_k(x) - f(x)\|_X < \epsilon,
\]
for every \( x \in E \). Put \( \varphi = \varphi_{k_0} \), then for every \( x \in E \), we have
\[
\|f(x) - \varphi(x)\|_X \leq \|f(x) - f_{k_0}(x)\|_X + \|f_{k_0}(x) - \varphi(x)\|_X < 2\epsilon.
\]
That means \( f \) is regulated on \( E \). \( \square \)

Based on Theorem 3.16, we develop two convergence theorems in Theorem 3.17.

**Theorem 3.17.** Let \( \{f_k\} \) be a sequence of regulated functions from a cell \( E \subseteq \mathbb{R}^n \) to a Hilbert space \( X \) and \( g \in BV(E, X) \). If \( \{f_k\} \) uniformly converges to a function \( f : E \to X \) on \( E \), then \( f \) is Henstock-Stieltjes integrable with respect to \( g \) on \( E \). Furthermore,
\[
\lim_{k \to \infty} (HS) \int_E \langle f_k, \, dg \rangle = (HS) \int_E \lim_{k \to \infty} \langle f_k, \, dg \rangle.
\]

**Proof.** By Theorem 3.16, \( f \) is regulated on \( E \), then by Theorem 3.15, \( f \) and \( f_k \) are Henstock-Stieltjes integrable with respect to \( g \) on \( E \) for every \( k \). For every \( k \in \mathbb{N} \), put \( A_k = (HS) \int_E \langle f_k, \, dg \rangle \). We can prove that \( \{A_k\} \) is a Cauchy sequence in \( \mathbb{R} \). Therefore, there exists \( A \in \mathbb{R} \), such that \( \{A_k\} \) converges to \( A \). The number \( A \) is the Henstock-Stieltjes integral value of \( f \) with respect to \( g \) on \( E \). Moreover, we have
\[
\lim_{k \to \infty} (HS) \int_E \langle f_k, \, dg \rangle = (HS) \int_E \lim_{k \to \infty} \langle f_k, \, dg \rangle.
\]
\( \square \)

In the next discussion, we define interval function from its corresponding point function to guarantee the integrability of the function \( f \) with respect to \( g \) on a cell \( E \), where \( f \) and \( g \) are step functions on \( E \). From the inner product on \( X \), the integral value can be determined explicitly.

**Definition 3.18.** Let \( E = [a, b] \) be an interval in \( \mathbb{R}^n \), where \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, b_3, \ldots, b_n) \).
(i) Let $F : \mathcal{I}(E) \to X$ be an interval function. We define a corresponding point function $F : E \to X$ as follows. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$F(x) = \begin{cases} 
\theta, & x_i = a_i, \text{ for some } i, 1 \leq i \leq n \\
F([a, x]), & \text{otherwise} 
\end{cases}$$

(ii) Conversely, given a point function $F : E \to X$, we may define a corresponding interval function $F : \mathcal{I}(E) \to X$ as follows. Let $I = [\alpha, \beta]$ be an interval, with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. For $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $n(\gamma)$ denotes the number of terms in $\gamma$ where $\gamma_i = \alpha_i$. We define

$$F(I) = \sum_{n(\gamma)} (-1)^{\gamma} F(\gamma).$$

Based on the Definition 3.18 point (ii), $F(I) = \theta$, for every degenerate interval $I$.

The point $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ in the Definition 3.18 is a vertex of the interval $I$. That means, the definition of interval function from its corresponding point function is done based on the vertex point of an interval. As a special case, for $n = 1$ and $I = [a, b]$, the associated interval function $F$ is $F(I) = F(b) - F(a)$.

**Theorem 3.19.** Let $K \in X$. If $f : E \to X$ is a function on $E$ and $g : E \to X$ is a function, where $g(x) = K$, for every $x \in E$, then $(HS) \int_E \langle f, dg \rangle$ exists. Moreover, $(HS) \int_E \langle f, dg \rangle = 0$.

**Proof.** Based on Definition 3.18, for every interval $I \subseteq E$, $g(I) = \theta$. Therefore, for every positive function $\delta$ and for every $\delta$-fine $\mathcal{D} = \{(I, x)\}$ of $E$, we have $\langle \mathcal{D} \rangle \sum \langle f(x), g(I) \rangle = 0$. \hfill \Box

Based on Theorem 3.19, a bounded variation function is integrable with respect a constant function on a cell $E$.

**Definition 3.20.** Let $K \in X$ and $B \subseteq \mathbb{R}^n$. We define a function $\chi_{(K, B)} : E \to X$ as follows.

$$\chi_{(K, B)}(x) = \begin{cases} 
K, & x \in B \\
\theta, & x \notin B. 
\end{cases}$$

**Theorem 3.21.** Let $K, L \in X$ and $B, C \subseteq \mathbb{R}^n$ are cells. If $f, g : E \to \mathbb{R}$ are functions, where $f(x) = \chi_{(L, C)}(x)$ and $g(x) = \chi_{(K, B)}(x)$, for every $(x) \in E$, then $(HS) \int_E \langle f, dg \rangle$ exists.

**Proof.** (1) Case $L \neq K$. 


\[
\chi_{(K,B)}(x) = \begin{cases} 
K, & x \in B \\
\theta, & x \notin B.
\end{cases}
\]

Therefore, for every interval \(I = [\alpha, \beta] \subseteq E\),
\[
g(I) = \begin{cases} 
K, & n_B(I) \text{ is odd and } n(\gamma_I) \text{ is even} \\
-K, & n_B(I) \text{ and } n(\gamma_I) \text{ are odd} \\
\theta, & n_B(I) \text{ is even},
\end{cases}
\]

where \(n_B(I)\) denotes the number of \(i (1 \leq i \leq n)\) such that \(\gamma_i = \alpha_i\) on the vertex of \(\gamma_I = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in B\) and \(n(\gamma)\) denotes the number \(i\) such that \(\gamma_i = \alpha_i\), respectively.

Let \(\delta\) be an arbitrary positive function on \(E\) such that
\[
\begin{cases} 
B(x, \delta(x)) \subseteq B, & \text{if } x \in B \\
B(x, \delta(x)) \subseteq B_c, & \text{if } x \notin B.
\end{cases}
\]

As corollary, for every \(\delta\)-fine partition
\[
\mathcal{D} = \{(I, x)\} = \{(I_1, x_1), (I_2, x_2), \ldots, (I_p, x_p)\}
\]
of \(E\), there exists \(i (1 \leq i \leq p)\), with \(x_i \in \partial B\). Let \(r\) denote the number of vertex of \(I_i\) that lays in \(B^o\), with \((x_i, I_i) \in \mathcal{D}\) and \(x_i \in \partial B\). If \(x_i \in \partial B\), then \(r = 1\) or \(r\) is an even number that is less than or equal to \(2^n\). If \(r\) is an even number, then \(g(I_i) = \theta\). If \(r = 1\),
\[
g(I_i) = \begin{cases} 
K, & n(\gamma_{I_i}) \text{ is odd} \\
-K, & n_{I_i}(\gamma_{I_i}) \text{ is even}.
\end{cases}
\]

We consider three following cases.

(1.1) The first case: \(B = C\).

\[
(\mathcal{D}) \sum \langle f(x), g(I) \rangle = \sum_{x_i \in \partial B, n_B(I_i) \text{ is odd}} \langle f(x_i), g(I_i) \rangle
\]

\[
= \sum_{x_i \in \partial B, n_B(I_i) \text{ is odd}} \langle L, (-1)^{n(\gamma_{I_i})}K \rangle
\]

The value of \(n(\gamma_{I_i})\) in (3) is done based on the number of vertices of \(B\) that lay in \(E^o\). We define \(q\) as number of vertices of \(B\) in \(E^o\). Since \(B\) is interval, if \(q\) is odd, then \(q = 1\). Therefore,
(i) If \( q \) is even, then the summation in (3) is equal
\[
\frac{q}{2} \langle L, K \rangle + \frac{q}{2} \langle L, -K \rangle = 0.
\]
So, \((\mathcal{D}) \sum (f(x), g(I)) = 0\). That means,
\[
\int_E \langle f, dg \rangle = 0.
\]
(ii) If \( q = 1 \), then \( n(\gamma_i) \) is even, \( n(\gamma_i) \) is odd, and \( |n(\gamma_i) - n(\gamma_i)| = 1 \).
Therefore, the value of \( \int_E \langle f, dg \rangle \) is \( \langle L, K \rangle \) either \( -\langle L, K \rangle \),
depends on the position \( B \) with respect to \( E \).

(1.2) Case \( B \subset C \).
That means \( \partial B \subset C \). See the proof of case in (1.1).

(1.3) Case \( C \subset B \).
If \( C \subset B^o \), the value \( f(x) = \theta \), for \( x \in \partial B \). As corollary,
\((\mathcal{D}) \sum (f(x), g(I)) = 0\). If there is \( y \in C \cap \partial B \), then
\[(\mathcal{D}) \sum \langle f(x), g(I) \rangle = \sum_{x_i \in \partial B, nB(I_i) \text{ is odd}, f(x_i) \in C} \langle f(x_i), g(I_i) \rangle \]
\[
= \sum_{x_i \in \partial B, nB(I_i) \text{ is odd}, f(x_i) \in C} \langle \theta, (-1)^{n(\gamma_i)} K \rangle = 0.
\]
Following the proof of case (2.1), \((HS) \int_E \langle f, dg \rangle \) has exactly one value, 0 or \( \langle L, K \rangle \) or \( -\langle L, K \rangle \).

(2) Case \( L = K \). In all cases, \( B = C \), \( B \subset C \), and \( C \subset B \), the proofs follow the proofs of Case (1) by replacing \( L \) with \( K \).

\( \square \)

Based on Theorem 3.21, if \( f \) and \( g \) have same points of discontinuity, then
\[ \int_E \langle f, dg \rangle \] exists, i.e. \( f \) is Henstock-Stieltjes integrable with respect \( g \) on \( E \).
This is a generalization on \([a, b] [3]\] and on a cell \( E \subset \mathbb{R}^n [5] \).

Every step function \( \varphi \) on a cell \( E \) induces a partition \( \{I_i : i = 1, 2, \ldots, p\} \) of \( E \), therefore \( \varphi \) can be written as summation of \( \chi_{(K_i, I_i)} \), \( i = 1, 2, 3, \ldots, p \).

**Theorem 3.22.** If \( f, g : E \to \mathbb{R} \) are step functions, then \((HS) \int_E \langle f, dg \rangle \) exists.

**Proof.** Let \( f = \sum_{i=1}^p \chi_{(K_i, I_i)} \) and \( g = \sum_{j=1}^s \chi_{(L_j, J_j)} \). From basic properties of the Henstock-Stieltjes integral and by Theorem 3.21, \((HS) \int_E \langle f, dg \rangle \) exists and its integral value is
\[
\sum_{i=1}^p \int_E \langle \chi_{(K_i, I_i)}, dg \rangle.
\]

\( \square \)
4. CONCLUDING REMARKS

By defining the integral as a real number, we have proved the properties of the Henstock integral still could be generalized. Moreover, the integrability of regulated function from a cell \( E \subseteq \mathbb{R}^n \) to a Hilbert space \( X \) with respect to a bounded variation function (Theorem 3.15) and some convergence theorems could be developed in to the integral (Theorem 3.16 and 3.17).

Furthermore, by using interval-function defined by its corresponding point-function, it can be seen the explicit formula in integrating the characteristic function \( f \) with respect to characteristic function \( g \) on a cell \( E \) Theorem 3.21.

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