Subdividing of Holder’s Inequalities on Time Scales via the Theory of Isotonic Linear Functionals

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Abstract

The aim of this paper is to establish some extensions of subdividing of Holder’s inequality and Minkowski’s inequality inequality to isotonic linear functionals taking into account that the time scale Cauchy delta, Cauchy nabla, α-diamond, multiple Riemann, and multiple Lebesque integrals all are isotonic linear functionals.

1. Introduction

In this paper we adopt the notations from the monograph [3] of Bohner and Peterson. For further information concerning time scales, see [3]. The following results will be useful below in order to establish the main results of this paper, and can be found in [3], in [12] and in [2]. The following two results present two important properties of the time scale Cauchy delta integrals.
Lemma 1. ([6], Corollary 3.3) If \( f \) is \( \Delta \)-integrable on \([a, b)\) then for an arbitrary positive number \( \alpha \) the function \(|f|^\alpha\) is \( \Delta \)-integrable on \([a, b)\).

Lemma 2. ([6], Theorem 3.6) Let \( f \) and \( g \) be \( \Delta \)-integrable functions on \([a, b)\). then their product \( fg \) is \( \Delta \)-integrable on \([a, b)\).

In the following we need to recall Holder’s inequality on time scales and two refinements of them which will be used below.

Lemma 3. ([3], p. 259, Theorem 6.13) Let \( a,b \in \mathbb{T} \). If \( f,g \in C_{rd}(\mathbb{T}, \mathbb{R})\) then
\[
\int_a^b |f(x)g(x)|\Delta x \leq \left[ \int_a^b |f(x)|^p \Delta x \right]^{\frac{1}{p}} \left[ \int_a^b |g(x)|^q \Delta x \right]^{\frac{1}{q}},
\]
where \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Lemma 4. ([12], Theorem 5) Let \( f,g,h \in C_{rd}([a, b], \mathbb{R}) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) with \( p > 1 \); then
\[
\left( \int_a^b |h(x)||f(x)|^p \Delta x \right)^{\frac{1}{p}} \left( \int_a^b |h(x)||g(x)|^q \Delta x \right)^{\frac{1}{q}} \geq \int_a^b |h(x)||f(x)g(x)|\Delta x.
\]

Lemma 5. ([12], Theorem 6) Let \( f,g,h \in C_{rd}([a, b], \mathbb{R}) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) with \( p < 0 \) or \( q < 0 \); then
\[
\left( \int_a^b |h(x)||f(x)|^p \Delta x \right)^{\frac{1}{p}} \left( \int_a^b |h(x)||g(x)|^q \Delta x \right)^{\frac{1}{q}} \leq \int_a^b |h(x)||f(x)g(x)|\Delta x.
\]

The following definition is given in [2], [5] and it is necessary to recall it here.

Definition 1. Let \( E \) be a nonempty set and \( L \) be a class of real-valued functions \( f : E \to \mathbb{R} \) having the following properties:

(L1) If \( f,g \in L \) and \( a,b \in \mathbb{R} \), then \((af + bg) \in L\).

(L2) If \( f(t) = 1 \) for all \( t \in E \), then \( f \in L\).

An isotonic linear functional is a functional \( A : L \to \mathbb{R} \) having the following properties:

(A1) If \( f,g \in L \) and \( a,b \in \mathbb{R} \), then \( A(af + bg) = aA(f) + bA(g)\).

(A2) If \( f \in L \) and \( f(t) \geq 0 \) for all \( t \in E \), then \( A(f) \geq 0\).

Now we will recall the classical Holder’s inequality for isotonic linear functionals as it appears in [8].
Theorem 1. ([2]) Let $E, L$, and $A$ such that $(L1)$, $(L2)$, $(A1)$ and $(A2)$ are satisfied. For $p \neq 1$, define $q = \frac{p}{p-1}$. Assume $|w||f|^p$, $|w||g|^q$, $|wfg| \in L$. If $p > 1$, then

\[ A(|wfg|) \leq A^\frac{1}{p}(|w||f|^p)A^\frac{1}{q}(|w||g|^q). \]

Then inequality is reversed if $0 < p < 1$ and $A(|w||g|^q) > 0$, and it is also reversed if $p < 0$ and $A(|w||f|^p) > 0$.

2. Subdividing of Holder’s inequalities on time scales

The following result is a subdividing of Holder’s inequality given on time scales as an analogue of Theorem 1.2 from [11].

Theorem 2. Let $s, t \in \mathbb{R}$, $p = \frac{s-t}{1-t}$, and $q = \frac{s-t}{s-1}$. We consider $a, b \in T$ and $f, g, h \in C_{rd}([a, b], \mathbb{R})$.

(i) If $s < 1 < t$ or $s > 1 > t$, then

\[
\int_a^b |h(x)||f(x)g(x)|\Delta x \leq \left( \int_a^b |h(x)||f(x)|^{sp}\Delta x \right)^{\frac{1}{p^2}} \cdot \left( \int_a^b |h(x)||g(x)|^{tq}\Delta x \right)^{\frac{1}{q^2}} \cdot \left( \int_a^b |h(x)||f(x)|^{tp}\Delta x \cdot \int_a^b |h(x)||g(x)|^{sq}\Delta x \right)^{\frac{1}{pq}}.
\]

(ii) If $s > t > 1$ or $1 < s > t$, then

\[
\int_a^b |h(x)||f(x)g(x)|\Delta x \geq \left( \int_a^b |h(x)||f(x)|^{sp}\Delta x \right)^{\frac{1}{p^2}} \cdot \left( \int_a^b |h(x)||g(x)|^{tq}\Delta x \right)^{\frac{1}{q^2}} \cdot \left( \int_a^b |h(x)||f(x)|^{tp}\Delta x \cdot \int_a^b |h(x)||g(x)|^{sq}\Delta x \right)^{\frac{1}{pq}}.
\]

Proof. (i) Taking into account that hypothesis $s < 1 < t$ or $s > 1 > t$ implies $p = \frac{s-t}{1-t} > 1$, $q = \frac{s-t}{s-1}$ and Lemma 1, by using Holder’s inequality on time scales we have

\[
\int_a^b |h(x)||f(x)g(x)|\Delta x = \int_a^b [|h(x)||fg(x)|^t]^{\frac{1}{t-1}} \cdot [(fg(x))^t]^{\frac{s-1}{t-1}} \Delta x \leq \frac{1}{t-1} \int_a^b |h(x)||fg(x)|^t \Delta x \cdot \int_a^b |h(x)||fg(x)|^t \Delta x.
\]

As in [11], using again Holder’s inequality from Lemma 4 for $\frac{s-t}{1-t} > 1$ and Lemma 1 we get,

\[
\int_a^b |h(x)||fg(x)|^s \Delta x \leq \left( \int_a^b |h(x)||f(x)|^{\frac{s-1}{t-1}} \Delta x \right)^{\frac{1}{t-1}} \cdot \left( \int_a^b |h(x)||g(x)|^{\frac{s-1}{s-1}} \Delta x \right)^{\frac{1}{s-1}}.
\]
and
\[ \int_a^b |h(x)||f(x)||g(x)||^t \Delta x \leq \left( \int_a^b |h(x)||f(x)|^{\frac{s}{s-1}} \Delta x \right)^{\frac{1}{s-1}} \left( \int_a^b |g(x)|^{\frac{t}{t-1}} \Delta x \right)^{\frac{1}{t-1}}. \]

Conclusion from (i) holds from last three inequalities.

(ii) From \( s > t > 1 \) or \( s < t < 1 \) we have \( \frac{s-1}{s-1} < 0 \) and \( t > s > 1 \) or \( t < s < 1 \) involves \( 0 < \frac{s-1}{s-1} < 1 \). Using now Hölder’s inequality from Lemma 5 for \( 0 < \frac{s-1}{s-1} < 1 \) or \( \frac{s-1}{s-1} < 0 \), we find
\[ \int_a^b |h(x)||f(x)||g(x)||^t \Delta x \geq \left[ \left( \int_a^b |h(x)||f(x)|^{\frac{s}{s-1}} \Delta x \right)^{\frac{1}{s-1}} \left( \int_a^b |g(x)|^{\frac{t}{t-1}} \Delta x \right)^{\frac{1}{t-1}} \right] \cdot \left[ \left( \int_a^b |h(x)||f(x)|^{\frac{s}{s-1}} \Delta x \right)^{\frac{1}{s-1}} \left( \int_a^b |g(x)|^{\frac{t}{t-1}} \Delta x \right)^{\frac{1}{t-1}} \right]. \]

3. Subdividing of Hölder’s inequalities for isotonic linear functionals

Starting from results given in [4], in [11], in [10] and in [1] we can state the following inequalities for isotonic linear functionals.

**Theorem 3.** Let \( s, t \in \mathbb{R}, p = \frac{s-1}{s-1}, \) and \( q = \frac{s-1}{s-1} \). Let \( L \) satisfying the conditions \( L1, L2 \) and \( A \) satisfying the conditions \( A1, A2 \) on the set \( E \). We assume \( |w||f|^p, |w||f|^q, |w||g|^q, |w||g|^q, |wfg|^p, |wfg|^q, |wfg|^t \) \( \in L \).

(i) If \( s < t < 1 \) or \( s > 1 > t \), then
\[ A(|wfg|) \leq A^\frac{1}{s-1}(|w||f|^p)A^\frac{1}{t-1}(|w||g|^q) \cdot [A(|w||f|^p)A(|w||g|^q)]^{\frac{1}{s-1}}. \]

(ii) If \( s > t > 1 \) or \( s < 1 < t \); \( t > s > 1 \) or \( t < s < 1 \), then
\[ A(|wfg|) \geq A^\frac{1}{s-1}(|w||f|^p)A^\frac{1}{t-1}(|w||g|^q) \cdot [A(|w||f|^p)A(|w||g|^q)]^{\frac{1}{s-1}}, \]
when \( A(|w||f|^p) > 0, A(|w||f|^q) > 0, A(|w||f|^p) > 0, A(|w||f|^q) > 0, A(|w||g|^q) > 0, A(|w||g|^q) > 0. \)

**Proof.** (i) By inequality (4) from Theorem 1, applied for \( p = \frac{s-1}{s-1} > 1, q = \frac{s-1}{s-1} \) we have
\[ A(|wfg|) = A \left( |wfg|^\frac{s-1}{s-1} \right) \leq A^\frac{s}{s-1}(|wfg|^p) \cdot A^\frac{t}{t-1}(|wfg|^q). \]

Applying again Theorem 1 for \( \frac{s-1}{s-1} > 1 \) we get
\[ A(|w||f|^p) \leq A^\frac{s}{s-1}(|w||f|^p)A^\frac{t}{t-1}(|w||g|^q). \]
and
\[ A(|w||fg|^t) \leq A^{\frac{1}{p-1}}(|w||f|^t)^{\frac{t}{p-1}} A^{\frac{1}{p-1}}(|w||g|^t)^{\frac{t}{p-1}}. \]

Taking into account these three inequalities we obtain the desired inequality.

For (ii) we use similary motivation and the reverse inequality from Theorem 1.

\[ \square \]

Taking into account Remark 2.5 from [4], we can state the following improvements of Minkowski’s inequality for isotonic linear functionals.

**Theorem 4.** (i) Let \( p > 0, s,t \in \mathbb{R} - \{0\} \), and \( s \neq t \). We consider \( p,s,t \in \mathbb{R} \) different numbers, such that \( s,t > 1, \frac{s-1}{p-t} > 1 \), \( L \) satisfy conditions \( L1, L2 \) and \( A \) satisfy \( A1, A2 \) on the set \( E \). If \( w,f,g \geq 0 \) on \( E \) with \( w(f + g)^p \), \( w(f + g)^s \), \( w(f + g)^t \), \( wf^s \), \( wg^s \), \( wf^t \), \( wg^t \in E \) then

\[ A(w(f + g)^p) \leq [A^{\frac{1}{p}}(wf^s) + A^{\frac{1}{p}}(wg^s)]^{\frac{t}{p}} \cdot [A^{\frac{1}{p}}(wf^t) + A^{\frac{1}{p}}(wg^t)]^{\frac{s}{p}}. \]

(ii) Let \( p > 0, s,t \in \mathbb{R} - \{0\} \), and \( s \neq t \). If we consider now \( p,s,t \in \mathbb{R} \) different numbers, such that \( s,t < 1, \frac{s-1}{p-t} < 1 \), \( L \) satisfy conditions \( L1, L2 \) and \( A \) satisfy conditions \( A1, A2 \) on the set \( E \) and if \( w,f,g \geq 0 \) on \( E \) with \( w(f + g)^p \), \( w(f + g)^s \), \( w(f + g)^t \), \( wf^s \), \( wg^s \), \( wf^t \), \( wg^t \in E \) then

\[ A(w(f + g)^p) \geq [A^{\frac{1}{p}}(wf^s) + A^{\frac{1}{p}}(wg^s)]^{\frac{t}{p}} \cdot [A^{\frac{1}{p}}(wf^t) + A^{\frac{1}{p}}(wg^t)]^{\frac{s}{p}}. \]

In this case we need the additional conditions \( A(w(f + g)^s) > 0, A(w(f + g)^t) > 0, A(wf^s) > 0, A(wg^s) > 0, A(wf^t) > 0, A(wg^t) > 0 \).

**Proof.** (i) We will use first Holder’s inequality, Theorem 1, page 136 and then Minkowski’s inequality, Theorem 2, on the same page, from [7]. For Holder’s inequality we use indices \( \frac{s-1}{p-t} \) and \( \frac{s-1}{s-p} \) obtaining:

\[ A(w(f + g)^p) = A(w(f + g)^p)^{\frac{p-t}{s-p}} (f + g)^{\frac{s-p}{s-p}} \leq A^{\frac{p-t}{s-p}}(w(f + g)^s) A^{\frac{s-p}{s-p}}(w(f + g)^t). \]

We use first time \( s > 1 \) for Minkowski’s inequality and the second time \( t > 1 \), obtaining:

\[ A^{\frac{s-1}{p-t}}(w(f + g)^s) A^{\frac{s-1}{s-p}}(w(f + g)^t) \leq [A^{\frac{1}{p}}(wf^s) + A^{\frac{1}{p}}(wg^s)]^{\frac{t}{s-p}} \cdot [A^{\frac{1}{p}}(wf^t) + A^{\frac{1}{p}}(wg^t)]^{\frac{s}{s-p}}. \]

(ii) We will use the same reason like before.

\[ \square \]

We continue by giving a refinement of the subdividing of Holder’s inequality from Theorem 3 for isotonic linear functionals, but first we enunciate Theorem 2.2 from [1] in the case of these functionals.
Theorem 5. Let \( 1 < p < \infty \) and let \( q = \frac{p}{p-1} \) be its conjugate exponent, \( L \) satisfy conditions \( L1, L2 \) and \( A \) satisfy \( A1, A2 \) on the set \( E \). If \(|f|^p, |g|^q, |f|g, |f|^2|g|^{2} \in L \). and if \( 1 < p \leq 2 \), then

\[
A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q) \left[ 1 - \frac{2}{p} \left( 1 - \frac{A(\|f\|^2|g|^{2})}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \right) \right] \leq A(\|fg\|) \leq A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q) \left[ 1 - \frac{2}{q} \left( 1 - \frac{A(\|f\|^2|g|^{2})}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \right) \right]
\]

while if \( 2 \leq p < \infty \), the terms \( \frac{1}{p} \) and \( \frac{1}{q} \) exchange their positions in the preceding inequalities.

Proof. We take in Lemma 2.1( [1]) \( u = \frac{|f|}{A^{1/2}(\|f\|^p)} \) and \( v = \frac{|g|}{A^{1/2}(\|g\|^q)} \) and by replacing in inequality \( \frac{1}{q}(u^{2} - v^{2})^{2} \leq \frac{w^{2}}{p} + \frac{v^{2}}{q} - uv \leq \frac{1}{p}(u^{2} - v^{2})^{2} \) we obtain

\[
\frac{1}{q} \left( \frac{|f|^{2}}{A^{1/2}(\|f\|^p)} - \frac{|g|^{2}}{A^{1/2}(\|g\|^q)} \right)^{2} \leq \frac{|f|^{p}}{pA(\|f\|^p)} + \frac{|g|^{q}}{qA(\|g\|^q)} - \frac{|f||g|}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \leq \frac{-1}{p} \left( \frac{|f|^{2}}{A^{1/2}(\|f\|^p)} - \frac{|g|^{2}}{A^{1/2}(\|g\|^q)} \right)^{2},
\]

or

\[
\frac{1}{q} \left( \frac{|f|^{p}}{A(\|f\|^p)} + \frac{|g|^{q}}{A(\|g\|^q)} - \frac{2|f|^{2}|g|^{2}}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \right) \leq \frac{|f|^{p}}{pA(\|f\|^p)} + \frac{|g|^{q}}{qA(\|g\|^q)} - \frac{2|f|^{2}|g|^{2}}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)}.
\]

Using hypothesis and condition A2 we have:

\[
\frac{1}{q} A \left( \frac{|f|^{p}}{A(\|f\|^p)} + \frac{|g|^{q}}{A(\|g\|^q)} - \frac{2|f|^{2}|g|^{2}}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \right) \leq \frac{A(\|f\|^p)}{pA(\|f\|^p)} + \frac{A(\|g\|^q)}{qA(\|g\|^q)} - \frac{A(\|fg\|)}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \leq \frac{1}{p} A \left( \frac{|f|^{p}}{A(\|f\|^p)} + \frac{|g|^{q}}{A(\|g\|^q)} - \frac{2|f|^{2}|g|^{2}}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \right)
\]

or by calculus,

\[
\frac{2}{q} \left( 1 - \frac{A(\|f|^{2}|g|^{2})}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \right) \leq 1 - \frac{A(\|fg\|)}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \leq \frac{2}{p} \left( 1 - \frac{A(\|f|^{2}|g|^{2})}{A^{1/2}(\|f\|^p)A^{1/2}(\|g\|^q)} \right)
\]

which leads to our conclusion. \( \Box \)
Theorem 6. Let \( s, t \in \mathbb{R} \), \( p = \frac{s+t}{s-t} \), and \( q = \frac{s+t}{s-t} \) such that \( s < 1 < t \) or \( s > 1 > t \), and \( L \) satisfy conditions \( L_1, L_2 \) and \( A \) satisfy conditions \( A_1, A_2 \) on the set \( E \). If \( f^{sp}, f^{tp}, g^{sq}, f^{tq}, f, g, (fg)^t, (fg)^s, f^{2p}g^{2q}, f^{2q}g^{2p}, (fg)^{\frac{s+t}{s-t}} \in L \) and \( f, g \) are positive functions then

\[
A(fg) \leq A^{\frac{1}{p}}(f^{sp})A^{\frac{1}{q}}(g^{sq}) \left[ A(f^{tp})A(g^{tq}) \right]^{\frac{1}{mp}} \left[ 1 - 2 \min \left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(\frac{f^{2p}g^{2q}}{f^{2q}g^{2p}})}{A^{\frac{1}{2}}(f^{sp})A^{\frac{1}{2}}(g^{tq})} \right) \right]^\frac{1}{p}.
\]

and

\[
A(fg) \geq A^{\frac{1}{p}}(f^{sp})A^{\frac{1}{q}}(g^{sq}) \left[ A(f^{tp})A(g^{tq}) \right]^{\frac{1}{mp}} \left[ 1 - 2 \max \left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(\frac{f^{2p}g^{2q}}{f^{2q}g^{2p}})}{A^{\frac{1}{2}}(f^{sp})A^{\frac{1}{2}}(g^{tq})} \right) \right]^\frac{1}{p}.
\]

Proof. By inequality given in Theorem 5, and applied for \( p = \frac{s+t}{s-t} > 1 \), \( q = \frac{s+t}{s-t} \) we have

\[
A(fg) = A \left( \frac{(fg)^t}{(fg)^s} \right)^{\frac{1}{s-t}} \left( \frac{(fg)^t}{(fg)^s} \right)^{\frac{1}{s-t}} \leq \frac{(fg)^t}{(fg)^s} \cdot A^{\frac{1}{s-t}}(f^{tp})A^{\frac{1}{s-t}}(g^{tq}) \left[ 1 - 2 \min \left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(\frac{f^{2p}g^{2q}}{f^{2q}g^{2p}})}{A^{\frac{1}{2}}(f^{sp})A^{\frac{1}{2}}(g^{tq})} \right) \right].
\]

Applying again Theorem 5 for \( \frac{s+t}{s-t} > 1 \) we get

\[
A((fg)^t) \leq A^{\frac{1}{s-t}}(f^{sp})A^{\frac{1}{s-t}}(g^{sq}) \left[ 1 - 2 \min \left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(\frac{f^{2p}g^{2q}}{f^{2q}g^{2p}})}{A^{\frac{1}{2}}(f^{sp})A^{\frac{1}{2}}(g^{tq})} \right) \right]
\]

and

\[
A((fg)^s) \leq A^{\frac{1}{s-t}}(f^{tp})A^{\frac{1}{s-t}}(g^{tq}) \left[ 1 - 2 \min \left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(\frac{f^{2p}g^{2q}}{f^{2q}g^{2p}})}{A^{\frac{1}{2}}(f^{sp})A^{\frac{1}{2}}(g^{tq})} \right) \right].
\]

Taking into account these three inequalities we obtain the desired inequality.

For second inequality we taking into account the first inequality from Theorem 5 and use the same reason like before. \( \square \)
Remark 1. (i) It is known that the time-scale integral is an isotonic linear functional as is given in Definition 1. Multiple Riemann delta time-scale integral is also an isotonic linear functional, see Theorem 3.6, [2].

(ii) The multiple Lebesque delta time-scale integral is also an isotonic linear functional, see Theorem 3.7, [2].

Therefore these inequalities from Theorem 3, Theorem 4, Theorem 5 and Theorem 6 can be rewritten for these kind of specific isotonic linear functionals.

We can give below an improvement of Holder’s inequality like in [1] for integral n time scales and then a refinement of Theorem 3, the subdividing of Holder’s inequality for integrals on time scales.

Remark 2. (i) Let $a, b \in \mathbb{T}$. If $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, are two positive functions then

$$
\left[ \int_{a}^{b} f(x)^p \Delta x \right]^\frac{1}{p} \left[ \int_{a}^{b} g(x)^q \Delta x \right]^\frac{1}{q} \cdot \left[ 1 - \frac{2}{\min\{p, q\}} \left( 1 - \frac{\int_{a}^{b} f(x)^\frac{p}{2} g(x)^\frac{q}{2} \Delta x}{(\int_{a}^{b} f(x)^p \Delta x)(\int_{a}^{b} g(x)^q \Delta x)^{\frac{1}{2}}} \right) \right] \leq \int_{a}^{b} f(x)g(x) \Delta x \leq \left[ \int_{a}^{b} f(x)^p \Delta x \right]^\frac{1}{p} \left[ \int_{a}^{b} g(x)^q \Delta x \right]^\frac{1}{q} \cdot \left[ 1 - \frac{2}{\max\{p, q\}} \left( 1 - \frac{\int_{a}^{b} f(x)^\frac{p}{2} g(x)^\frac{q}{2} \Delta x}{(\int_{a}^{b} f(x)^p \Delta x)(\int_{a}^{b} g(x)^q \Delta x)^{\frac{1}{2}}} \right) \right]
$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) Let $s, t \in \mathbb{R}$, $p = \frac{s+1}{s-1}$, and $q = \frac{t+1}{t-1}$. We consider $a, b \in \mathbb{T}$ and $f, g \in C_{rd}([a, b], \mathbb{R})$ two positive functions such that if $s < 1 < t$ or $s > 1 > t$, then

$$
\int_{a}^{b} f(x)g(x) \Delta x \leq \left( \int_{a}^{b} f(x)^s \Delta x \right)^\frac{1}{s} \cdot \left( \int_{a}^{b} g(x)^q \Delta x \right)^\frac{1}{q} \cdot \left( \int_{a}^{b} f(x)^{pt} \Delta x \cdot \int_{a}^{b} g(x)^{st} \Delta x \right)^\frac{1}{pt} \cdot \left[ 1 - 2 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_{a}^{b} f(x)^{mp} g(x)^n \Delta x}{(\int_{a}^{b} f(x)^p \Delta x)(\int_{a}^{b} g(x)^q \Delta x)^{\frac{1}{2}}} \right) \right]^{\frac{1}{2}}
$$

$$
\cdot \left[ 1 - 2 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_{a}^{b} f(x)^{mp} g(x)^n \Delta x}{(\int_{a}^{b} f(x)^p \Delta x)(\int_{a}^{b} g(x)^q \Delta x)^{\frac{1}{2}}} \right) \right]^{\frac{1}{q}}
$$

$$
\cdot \left[ 1 - 2 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_{a}^{b} f(x)^{mp} g(x)^n \Delta x}{(\int_{a}^{b} f(x)^p \Delta x)(\int_{a}^{b} g(x)^q \Delta x)^{\frac{1}{2}}} \right) \right]^{\frac{1}{4}}
$$

and

$$
\int_{a}^{b} f(x)g(x) \Delta x \geq \left( \int_{a}^{b} f(x)^s \Delta x \right)^\frac{1}{s} \cdot \left( \int_{a}^{b} g(x)^q \Delta x \right)^\frac{1}{q} \cdot \left( \int_{a}^{b} f(x)^{pt} \Delta x \cdot \int_{a}^{b} g(x)^{st} \Delta x \right)^\frac{1}{pt}\cdot \left[ 1 - 2 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_{a}^{b} f(x)^{mp} g(x)^n \Delta x}{(\int_{a}^{b} f(x)^p \Delta x)(\int_{a}^{b} g(x)^q \Delta x)^{\frac{1}{2}}} \right) \right]^{\frac{1}{2}}
$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. 

We can give below an improvement of Holder’s inequality like in [1] for integral n time scales and then a refinement of Theorem 3, the subdividing of Holder’s inequality for integrals on time scales.
Subdividing of Holder’s inequalities

\[
\left( \int_a^b f(x)^p \Delta x \cdot \int_a^b g(x)^q \Delta x \right)^{\frac{1}{pq}} \left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f(x)^p g(x)^q \Delta x}{\left( \int_a^b f(x)^p \Delta x \int_a^b g(x)^q \Delta x \right)^{\frac{1}{2}}} \right) \right]^{\frac{1}{p}} \\
\cdot \left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f(x)^p g(x)^q \Delta x}{\left( \int_a^b f(x)^p \Delta x \int_a^b g(x)^q \Delta x \right)^{\frac{1}{2}}} \right) \right]^{\frac{1}{q}} \\
+ \left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f(x)^p g(x)^q \Delta x}{\left( \int_a^b f(x)^p \Delta x \int_a^b g(x)^q \Delta x \right)^{\frac{1}{2}}} \right) \right]^{\frac{1}{p}} \\
+ \left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f(x)^p g(x)^q \Delta x}{\left( \int_a^b f(x)^p \Delta x \int_a^b g(x)^q \Delta x \right)^{\frac{1}{2}}} \right) \right]^{\frac{1}{q}} 
\]

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