Some Integral Inequalities via
the Theory of Isotonic Linear Functionals

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Abstract

The aim of this paper is to establish some extensions of Qi’s inequality to isotonic linear functionals taking into account that the time scale Cauchy delta, Cauchy nabla, α-diamond, multiple Riemann, and multiple Lebesque integrals all are isotonic linear functionals. Moreover several consequences of subdividing of Holder’s inequality were given in section 2, using that the time scale Cauchy nabla and α-diamond are all isotonic linear functionals.

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1. Introduction

In this paper we use the notations from the monograph [3] of Bohner and Peterson. For further information concerning time scales, see [3]. The following results will be useful below in order to establish the main results of this paper, and can be found in [3], in [15] and in [2]. Also in [8] appear other usual examples of isotonic linear functionals that are normalised. Therefore the results from this paper which take place for such functionals can be rewritten
for these particular examples. The following results present some important properties of the time scale Cauchy delta integrals.

**Lemma 1.** ([9], Corollary 3.3) If $f$ is $\Delta$-integrable on $[a,b)$ then for an arbitrary positive number $\alpha$ the function $|f|^\alpha$ is $\Delta$-integrable on $[a,b)$.

**Lemma 2.** ([9], Theorem 3.6) Let $f$ and $g$ be $\Delta$-integrable functions on $[a,b)$. then their product $fg$ is $\Delta$-integrable on $[a,b)$.

In the following it is necessary to recall Holder’s inequality on time scales and two refinements of them which will be used below in this paper.

**Lemma 3.** ([3], p. 259, Theorem 6.13) Let $a,b \in \mathbb{T}$. If $f,g \in C_{rd}(\mathbb{T}, \mathbb{R})$, then

\begin{equation}
\int_a^b |f(x) g(x)| \Delta x \leq \left[ \int_a^b |f(x)|^p \Delta x \right]^{\frac{1}{p}} \left[ \int_a^b |g(x)|^q \Delta x \right]^{\frac{1}{q}},
\end{equation}

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

**Lemma 4.** ([15], Theorem 5) Let $f,g,h \in C_{rd}([a,b], \mathbb{R})$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$; then

\begin{equation}
\left( \int_a^b |h(x)||f(x)|^p \Delta x \right)^{\frac{1}{p}} \left( \int_a^b |h(x)||g(x)|^q \Delta x \right)^{\frac{1}{q}} \geq \int_a^b |h(x)||f(x)g(x)| \Delta x.
\end{equation}

**Lemma 5.** ([15], Theorem 6) Let $f,g,h \in C_{rd}([a,b], \mathbb{R})$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p < 0$ or $q < 0$; then

\begin{equation}
\left( \int_a^b |h(x)||f(x)|^p \Delta x \right)^{\frac{1}{p}} \left( \int_a^b |h(x)||g(x)|^q \Delta x \right)^{\frac{1}{q}} \leq \int_a^b |h(x)||f(x)g(x)| \Delta x.
\end{equation}

The following important definition is given in [2], [7] and it is necessary to recall it here.

**Definition 1.** Let $E$ be a nonempty set and $L$ be a class of real-valued functions $f : E \to \mathbb{R}$ having the following properties:

- **(L1)** If $f, g \in L$ and $a, b \in \mathbb{R}$, then $(af + bg) \in L$.
- **(L2)** If $f(t) = 1$ for all $t \in E$, then $f \in L$.

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:

- **(A1)** If $f, g \in L$ and $a, b \in \mathbb{R}$, then $A(af + bg) = aA(f) + bA(g)$.
- **(A2)** If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

The mapping $A$ is said to be normalised if

- **(A3)** $A(1) = 1$. 
Now we will recall Holder’s inequality for isotonic linear functionals as it appears in [11].

Theorem 1. ([2]) Let $E, L, A$ such that (L1), (L2), (A1) and (A2) are satisfied. For $p \neq 1$, define $q = \frac{p}{p-1}$. Assume $|w||f|^p, |w||g|^q, |wfg| \in L$. If $p > 1$, then

\[
A(|wfg|) \leq A^\frac{1}{p}(|w||f|^{p})A^\frac{1}{q}(|w||g|^{q}).
\]

Then inequality is reversed if $0 < p < 1$ and $A(|w||g|^q) > 0$, and it is also reversed if $p < 0$ and $A(|w||f|^p) > 0$.

2. Subdividing of Holder’s inequalities on time scales

We enunciate a refinement of the subdividing of Holder’s inequality for isotonic linear functionals, see [5] for proof, or [6] and then we present several interesting consequences of this result.

Theorem 2. Let $s, t \in \mathbb{R}, p = \frac{s-t}{s-1}$, and $q = \frac{s-t}{s-1}$ such that $s < 1 < t$ or $s > 1 > t$, and $L$ satisfy conditions L1, L2 and $A$ satisfy conditions A1, A2 on the set $E$. If $f^{sp}, f^{tp}, g^{sq}, g^{tq}, fg, (fg)^t, (fg)^s, f^{\frac{sp}{2} + \frac{tq}{2}}, f^{\frac{tp}{2} + \frac{tq}{2}}, (fg)^{t+\frac{1}{2}} \in L$ and $f, g$ are positive functions then

\[
A(fg) \leq A^\frac{1}{p^2}(f^{sp})A^\frac{1}{q^2}(g^{tq}) \left[ A(f^{tp})A(g^{sq}) \right]^\frac{1}{p^2} \left[ 1 - 2 \min\left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(f^{\frac{sp}{2} + \frac{tq}{2}})}{A^\frac{1}{2}(f^{sp})A^\frac{1}{2}(g^{tq})} \right) \right]^\frac{1}{p}.
\]

\[
\cdot \left[ 1 - 2 \min\left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(f^{\frac{tp}{2} + \frac{tq}{2}})}{A^\frac{1}{2}(f^{tp})A^\frac{1}{2}(g^{tq})} \right) \right]^\frac{1}{q}.
\]

\[
\cdot \left[ 1 - 2 \min\left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A((fg)^{t+\frac{1}{2}})}{A^\frac{1}{2}((fg)^t)A^\frac{1}{2}((fg)^s)} \right) \right],
\]

and

\[
A(fg) \geq A^\frac{1}{p^2}(f^{sp})A^\frac{1}{q^2}(g^{tq}) \left[ A(f^{tp})A(g^{sq}) \right]^\frac{1}{p^2} \left[ 1 - 2 \max\left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(f^{\frac{sp}{2} + \frac{tq}{2}})}{A^\frac{1}{2}(f^{sp})A^\frac{1}{2}(g^{tq})} \right) \right]^\frac{1}{p}.
\]

\[
\cdot \left[ 1 - 2 \max\left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A(f^{\frac{tp}{2} + \frac{tq}{2}})}{A^\frac{1}{2}(f^{tp})A^\frac{1}{2}(g^{tq})} \right) \right]^\frac{1}{q}.
\]

\[
\cdot \left[ 1 - 2 \max\left\{ \frac{1}{q}, \frac{1}{p} \right\} \left( 1 - \frac{A((fg)^{t+\frac{1}{2}})}{A^\frac{1}{2}((fg)^t)A^\frac{1}{2}((fg)^s)} \right) \right].
\]
Remark 1. Using that the Cauchy nabla time-scales integral is an isotonic linear functional, see Theorem 3.4, [2], we can give the following subdividing of Holder’s inequalities for the Cauchy nabla time-scales integral:

Let $s, t \in \mathbb{R}$, $p = \frac{q'}{p'}$, and $q = \frac{q'}{q'}$. We consider $a, b \in \mathbb{T}$ and $f, g, h \in C_{ud}([a, b], \mathbb{R})$.

(a) If $s < 1 < t$ or $s > 1 > t$, then

$$
\int_a^b |h(x)||f(x)g(x)|\nabla x \leq \left( \int_a^b |h(x)||f(x)|^{sp}\nabla x \right)^{\frac{1}{p'}} \cdot \left( \int_a^b |h(x)||g(x)|^{sq}\nabla x \right)^{\frac{1}{q'}}
\cdot \left( \int_a^b |h(x)||f(x)|^{tp}\nabla x \cdot \int_a^b |h(x)||g(x)|^{sq}\nabla x \right)^{\frac{1}{p'}}.
$$

(b) If $s > t > 1$ or $1 < t < 1$, then

$$
\int_a^b |h(x)||f(x)g(x)|\nabla x \geq \left( \int_a^b |h(x)||f(x)|^{sp}\nabla x \right)^{\frac{1}{p'}} \cdot \left( \int_a^b |h(x)||g(x)|^{sq}\nabla x \right)^{\frac{1}{q'}}
\cdot \left( \int_a^b |h(x)||f(x)|^{tp}\nabla x \cdot \int_a^b |h(x)||g(x)|^{sq}\nabla x \right)^{\frac{1}{p'}}.
$$

(c) If $s < 1 < t$ or $s > 1 > t$, and $f, g$ are two positive functions then

$$
\int_a^b f(x)g(x)\nabla x \leq \left( \int_a^b f(x)^{sp}\nabla x \right)^{\frac{1}{p'}} \cdot \left( \int_a^b g(x)^{sq}\nabla x \right)^{\frac{1}{q'}}
\cdot \left( \int_a^b f(x)^{tp}\nabla x \cdot \int_a^b g(x)^{sq}\nabla x \right)^{\frac{1}{p'}}
\left[ 1 - 2 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f^\frac{p}{q}(x)g^\frac{q}{p}(x)\nabla x}{(\int_a^b f^{sp}(x)\nabla x \int_a^b g^{sq}(x)\nabla x)^{\frac{1}{2}}} \right)^{\frac{1}{2}} \right]^{\frac{1}{p'}}.
$$

and

$$
\int_a^b f(x)g(x)\nabla x \geq \left( \int_a^b f(x)^{sp}\nabla x \right)^{\frac{1}{p'}} \cdot \left( \int_a^b g(x)^{sq}\nabla x \right)^{\frac{1}{q'}}
\cdot \left( \int_a^b f(x)^{tp}\nabla x \cdot \int_a^b g(x)^{sq}\nabla x \right)^{\frac{1}{p'}}
\left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f^\frac{p}{q}(x)g^\frac{q}{p}(x)\nabla x}{(\int_a^b f^{sp}(x)\nabla x \int_a^b g^{sq}(x)\nabla x)^{\frac{1}{2}}} \right)^{\frac{1}{2}} \right]^{\frac{1}{p'}}.
$$
Some integral inequalities 2155

\[
\left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b (f(x)g(x))^{\frac{p+q}{2}} \, dx}{\left( \int_a^b (f(x)g(x))^s \, dx \int_a^b (f(x)g(x))^t \, dx \right)^{\frac{1}{2}}} \right)^{\frac{1}{p}} \right].
\]

**Remark 2.** Using that the Cauchy \( \alpha \)-diamond time scale integral is an isotonic linear functional, see Theorem 3.5, [2], we can give the following subdividing of Holder's inequalities for the Cauchy \( \alpha \)-diamond time-scales integral:

Let \( s, t \in \mathbb{R} \), \( p = \frac{s-t}{s-1} \), and \( q = \frac{s-t}{s-1} \). We consider \( a, b, t \in \mathbb{T} \) and \( f, g, h : [a, b] \to \mathbb{R} \) be three \( \diamond \)-integrable functions.

(i) If \( s < t < 1 \) or \( s > 1 > t \), then

\[
\int_a^b |h(x)|f(x)g(x)|_{\diamond} \leq \left( \int_a^b |h(x)||f(x)|^p_{\diamond} \right)^{\frac{1}{p}} \cdot \left( \int_a^b |h(x)||g(x)|^q_{\diamond} \right)^{\frac{1}{q}}.
\]

(ii) If \( s > t > 1 \) or \( s < 1 < t \); \( t > s > 1 \) or \( t < s < 1 \), then

\[
\int_a^b |h(x)||f(x)||g(x)|_{\diamond} \geq \left( \int_a^b |h(x)||f(x)|^p_{\diamond} \right)^{\frac{1}{p}} \cdot \left( \int_a^b |h(x)||g(x)|^q_{\diamond} \right)^{\frac{1}{q}}.
\]

(iii) If \( s < 1 < t \) or \( s > 1 > t \), and \( f, g \) are two positive functions then

\[
\int_a^b f(x)g(x)_{\diamond} \leq \left( \int_a^b f(x)^p_{\diamond} \right)^{\frac{1}{p}} \cdot \left( \int_a^b g(x)^q_{\diamond} \right)^{\frac{1}{q}}.
\]

\[
\cdot \left( \int_a^b f(x)^p_{\diamond} \cdot \int_a^b g(x)^q_{\diamond} \right)^{\frac{1}{pq}} \left[ 1 - 2 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f(x)^p_{\diamond} g(x)^q_{\diamond} \, dx}{\left( \int_a^b f(x)^p_{\diamond} \, dx \int_a^b g(x)^q_{\diamond} \, dx \right)^{\frac{1}{2}}} \right)^{\frac{1}{p}} \right].
\]

\[
\cdot \left[ 1 - 2 \min\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f(x)^p_{\diamond} g(x)^q_{\diamond} \, dx}{\left( \int_a^b f(x)^p_{\diamond} \, dx \int_a^b g(x)^q_{\diamond} \, dx \right)^{\frac{1}{2}}} \right)^{\frac{1}{q}} \right].
\]

and

\[
\int_a^b f(x)g(x)_{\diamond} \geq \left( \int_a^b f(x)^p_{\diamond} \right)^{\frac{1}{p}} \cdot \left( \int_a^b g(x)^q_{\diamond} \right)^{\frac{1}{q}}.
\]

\[
\cdot \left( \int_a^b f(x)^p_{\diamond} \cdot \int_a^b g(x)^q_{\diamond} \right)^{\frac{1}{pq}} \left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f(x)^p_{\diamond} g(x)^q_{\diamond} \, dx}{\left( \int_a^b f(x)^p_{\diamond} \, dx \int_a^b g(x)^q_{\diamond} \, dx \right)^{\frac{1}{2}}} \right)^{\frac{1}{p}} \right].
\]
\[
\cdot \left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b f^{\frac{p}{q}}(x)g^{\frac{q}{p}}(x) \circ \alpha x}{\left( \int_a^b f^{p}(x) \circ \alpha x \int_a^b g^{q}(x) \circ \alpha x \right)^{\frac{1}{2}}} \right)^{\frac{1}{q}} \right]^{1 - \frac{1}{q}} \cdot \left[ 1 - 2 \max\left\{ \frac{1}{p}, \frac{1}{q} \right\} \left( 1 - \frac{\int_a^b (f(x)g(x))^{\frac{1}{p} + \frac{1}{q}} \circ \alpha x}{\left( \int_a^b (f(x)g(x))^s \circ \alpha x \int_a^b (f(x)g(x))^t \circ \alpha x \right)^{\frac{1}{2}}} \right)^{\frac{1}{q}} \right]^{1 - \frac{1}{q}}.
\]

**Remark 3.** We can also consider Example 3.3, from [2] when \( T = \mathbb{R} \), when \( T = \mathbb{Z} \), and when \( T = h\mathbb{Z} \), \( h > 0 \) in Theorem 3.2, and to rewrite inequalities from Theorem 3 and 4.

According to [2], when the time scale is the set of all real numbers the time-scale integral is an ordinary integral, when the time-scale is the set of all integers the time-scale integral is a sum, when the time scale is the set of all integer powers of a fixed number the time-scale integral is a Jackson integral.

3. Several inequalities and Qi’s inequalities for isotonic linear functionals

Next results represent variants of several results given in [12] in Lemma 2.5, Lemma 2.6, Lemma 2.7, Theorem 3.1 and Theorem 3.3 in the case of isotonic linear functionals.

**Lemma 6.** Let \( E, L \) and \( A \) be such that \( L_1, L_2, A_1, A_2 \) are satisfied. If \( f, g, \frac{f^p}{g^q} \in L \) are positive functions then

\[
A \left( \frac{f^p}{g^q} \right) \geq \frac{A^p(f)}{A^q(g)},
\]

where \( p > 1 \) or \( p < 0 \) while \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** We apply Holder’s inequality from Theorem 1 when \( p > 1 \) and \( f, g, \frac{f^p}{g^q} \in L \) are positive functions, obtaining:

\[
A(f) = A \left( \frac{f}{g^{\frac{1}{q}}} \right) \leq A^{\frac{1}{p}} \left( \frac{f^p}{g^q} \right) A^{\frac{1}{q}}(g).
\]

Then we take the \( p \)-th power on both sides of the inequalities and have:

\[
A^p(f) \leq A \left( \frac{f^p}{g^q} \right) A^{\frac{p}{q}}(g).
\]

When \( p < 0 \) we take into account the inverse of Holder’s inequality, Theorem 1. \( \Box \)
Lemma 7. Let $E$, $L$ and $A$ be such that $L_1$, $L_2$, $A_1$, $A_2$ are satisfied. If $f, g, f^\frac{1}{p}, g^\frac{1}{q} \in L$ are positive on $E$ such that $m \leq \frac{f(x)}{g(x)} \leq M$ on $E$, where $m > 0$ and $M < \infty$ then we have

$$A^\frac{1}{p}(f)A^\frac{1}{q}(g) \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} A(f^\frac{1}{p}g^\frac{1}{q}),$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the hypothesis $\frac{f(x)}{g(x)} \leq M$, as in Lemma 2.6, [12], we find

$$f^\frac{1}{p}(x)g^\frac{1}{q}(x) \geq M^{-\frac{1}{q}}f^\frac{1}{p}(x)f^\frac{1}{p}(x) = M^{-\frac{1}{q}}f(x),$$
on $E$. Therefore by Definition 1, $A_2$ and $L_1$ we have

$$A(f^\frac{1}{p}g^\frac{1}{q}) \geq M^{-\frac{1}{q}}A(f) \geq 0 \text{ or } A^\frac{1}{q}(f^\frac{1}{p}g^\frac{1}{q}) \geq M^{-\frac{1}{pq}}A^\frac{1}{p}(f) \geq 0.$$

If we consider now $\frac{f(x)}{g(x)} \geq m$ by the same reason we find that

$$A(f^\frac{1}{p}g^\frac{1}{q}) \geq m^\frac{1}{q}A(g) \geq 0 \text{ or } A^\frac{1}{q}(f^\frac{1}{p}g^\frac{1}{q}) \geq m^\frac{1}{pq}A^\frac{1}{p}(g) \geq 0.$$

From inequalities (5) and (6) we obtain

$$A(f^\frac{1}{p}g^\frac{1}{q}) \geq \left(\frac{m}{M}\right)^{\frac{1}{pq}} A^\frac{1}{p}(f)A^\frac{1}{q}(g).$$

\[\Box\]

Theorem 3. Let $E$, $L$ and $A$ be such that $L_1$, $L_2$, $A_1$, $A_2$ are satisfied. If $f, f^p \in L$, $f$ is positive and $A(f) \geq A^{p-1}(1)$ then

$$A(f^p) \geq A^{p-1}(f).$$

Proof. By Lemma 6 and hypothesis we have,

$$A(f^p) = A\left(\frac{f^p}{1^{\frac{1}{p}}}\right) \geq \frac{A^p(f)}{A^\frac{1}{p}(1)} = \frac{A^{p-1}(f)A(f)}{A^{p-1}(1)} \geq A^{p-1}(f).$$

\[\Box\]

Consequence 1. Let $E$, $L$ and $A$ be such that $L_1$, $L_2$, $A_1$, $A_2$ and $A_3$ are satisfied. If $f, f^p \in L$, $f$ is positive and $A(f) \geq 1$ then

$$A(f^p) \geq A^p(f).$$

Remark 4. Let $E$, $L$ and $A$ be such that $L_1$, $L_2$, $A_1$, $A_2$ are satisfied. If $f^p, g^q, f^q g^p \in L$ are positive on $E$ such that $m \leq \frac{f^p(x)}{g^q(x)} \leq M$ on $E$, where $m > 0$ and $M < \infty$ then we have

$$A^\frac{1}{p}(f^p)A^\frac{1}{q}(g^q) \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} A(fg),$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. 

Theorem 4. Let $E$, $L$ and $A$ be such that $L1$, $L2$, $A1$, $A2$ are satisfied. If $f$, $f^\frac{1}{p}$, $f^p \in L$ are such that $m \leq f^p(x) \leq M$ on $E$, where $m > 0$ and $M < \infty$ then we have

$$A^\frac{1}{p}(f^p) \leq \left(\frac{M}{m}\right)^{\frac{2}{pq}} A^{-\frac{p+1}{q}}(1) A^p(f^\frac{1}{p}),$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking $g(x) = 1$ on $E$ in Remark 6 we get

$$A^\frac{1}{p}(f^p) A^\frac{1}{q}(1) \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} A(f) \quad \text{or} \quad A^\frac{1}{p}(f^p) \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} A^{-\frac{1}{q}}(1) A(f).$$

Now putting $g(x) = 1$ in Lemma 7 we will obtain

$$A^\frac{1}{p}(f) \leq A^{-\frac{1}{q}}(1) \left(\frac{M}{m}\right)^{\frac{1}{pq}} A(f^\frac{1}{p}) \quad \text{or} \quad A(f) \leq A^{-\frac{1}{q}}(1) \left(\frac{M}{m}\right)^{\frac{1}{pq}} A^p(f^\frac{1}{p})$$

using the hypothesis that $m^\frac{1}{p} \leq f(x) \leq M^\frac{1}{p}$. By using inequalities (7) and (8) we find

$$A^\frac{1}{p}(f^p) \leq \left(\frac{M}{m}\right)^{\frac{2}{pq}} A^{-\frac{1}{q} - \frac{p}{q}}(1) A^p(f^\frac{1}{p}).$$

□

Consequence 2. If in addition, $A$ is normalised i.e. $A3$ satisfies $A3$ then the previous inequality becomes,

$$A^\frac{1}{p}(f^p) \leq \left(\frac{M}{m}\right)^{\frac{2}{pq}} A^p(f^\frac{1}{p}).$$

Remark 5. The multiple Lebesque delta time-scale integral, the Cauchy nabla time-scales integral and the Cauchy $\alpha$-diamond time scale integral are also isotonic linear functionals, therefore these inequalities from Lemma 6, Lemma 7, Theorem 3 and Theorem 4 can be rewritten for these kind of special isotonic linear functionals.

Remark 6. (i) Let $f, g \in C{id}([a, b], \mathbb{R})$ be two positive functions. Then the following inequality holds:

$$\int_a^b f^p(x) \frac{\nabla x}{g^\frac{q}{p}(x)} \geq \left[\int_a^b (x) \nabla x\right]^p \frac{p}{\left[\int_a^b g(x) \nabla x\right]^{\frac{q}{p}}},$$

where $p > 1$ or $p < 0$ while $\frac{1}{p} + \frac{1}{q} = 1$. 
(ii) Let \( a, b \in \mathbb{T} \) and \( f, g : [a, b] \to \mathbb{R} \) be two \( \diamond \)-integrable functions. Then we have
\[
\int_a^b \frac{f^p(x)}{g^q(x)} \diamond x \geq \frac{[f^p_a(x) \diamond x]^{\frac{1}{p}}}{[f^q_a g(x) \diamond x]^{\frac{1}{q}}},
\]
where \( p > 1 \) or \( p < 0 \) while \( \frac{1}{p} + \frac{1}{q} = 1 \).

The following two results will help us to present a refinement of inequality from Theorem 3.

**Lemma 8.** Let \( E, L \) and \( A \) be such that \( L_1, L_2, A_1, A_2 \) are satisfied. If \( f, g, f^p, g^q \in L \) are positive functions then
\[
A(f^p) \left[ 1 - 2 \min \{ \frac{1}{p}, \frac{1}{q} \} \left( 1 - \frac{A(f^{\frac{p}{q}} g^{\frac{1}{q}})}{A_{\frac{1}{q}} \left( f^{\frac{p}{q}} A_{\frac{1}{q}}(g) \right)} \right) \right]^p \geq A^p(f - A^{p-1}(1)),
\]
where \( p > 1 \) or \( p < 0 \) while \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** We apply Holder’ s inequality from Theorem 5, see [6] when \( p > 1 \) and \( f, g, f^p, g^q \in L \) are positive functions, obtaining:
\[
A(f^p) \leq A^\frac{p}{q} \left( f^{\frac{p}{q}} A_{\frac{1}{q}}(g) \right) \left[ 1 - 2 \min \{ \frac{1}{p}, \frac{1}{q} \} \left( 1 - \frac{A(f^{\frac{p}{q}} g^{\frac{1}{q}})}{A_{\frac{1}{q}} \left( f^{\frac{p}{q}} A_{\frac{1}{q}}(g) \right)} \right) \right]^p.
\]

Then we take the \( p \)-th power on both sides of the inequalities and have:
\[
A^p(f) \leq A \left( f^p \right) \left[ 1 - 2 \min \{ \frac{1}{p}, \frac{1}{q} \} \left( 1 - \frac{A(f^{\frac{p}{q}} g^{\frac{1}{q}})}{A_{\frac{1}{q}} \left( f^{\frac{p}{q}} A_{\frac{1}{q}}(g) \right)} \right) \right]^p.
\]

**Theorem 5.** Let \( E, L \) and \( A \) be such that \( L_1, L_2, A_1, A_2 \) are satisfied. If \( f, f^p, f^{\frac{p}{q}} \in L \), \( f \) is positive and \( A(f) \geq A^{p-1}(1) \) then
\[
A(f^p) \left[ 1 - 2 \min \{ \frac{1}{p}, \frac{1}{q} \} \left( 1 - \frac{A(f^{\frac{p}{q}})}{A_{\frac{1}{q}}(f^p A_{\frac{1}{q}}(1))} \right) \right]^p \geq A^{p-1}(f).
\]

**Proof.** By Lemma 8 and hypothesis we have,
\[
A(f^p) \left[ 1 - 2 \min \{ \frac{1}{p}, \frac{1}{q} \} \left( 1 - \frac{A(f^{\frac{p}{q}})}{A_{\frac{1}{q}}(f^p A_{\frac{1}{q}}(1))} \right) \right]^p =
\]
\[
= A \left( \frac{f^p}{1^q} \right) \left[ 1 - 2 \min\{ \frac{1}{p}, \frac{1}{q} \} \left( 1 - \frac{A(f^\frac{p}{q})}{A^\frac{p}{q}(f^p)A^\frac{1}{q}(1)} \right) \right]^p \geq \frac{A^p(f)}{A^{\frac{p}{q}}(1)} = \frac{A^{p-1}(f)A(f)}{A^{p-1}(1)} \geq A^{p-1}(f).
\]

**Consequence 3.** Let \( E, L \) and \( A \) be such that \( L_1, L_2, A_1, A_2 \) and \( A_3 \) are satisfied. If \( f, f^p, f^{\frac{p}{q}} \in L, f \) is positive and \( A(f) \geq 1 \) then

\[
A(f^p) \left[ 1 - 2 \min\{ \frac{1}{p}, \frac{1}{q} \} \left( 1 - \frac{A(f^{\frac{p}{q}})}{A^{\frac{p}{q}}(f^p)} \right) \right]^p \geq \frac{A^p(f)}{A^{\frac{p}{q}}(1)} \geq A^{p-1}(f).
\]

Next results present some improvements of some integral inequalities given by Qi and Yin, [12], in the cases of delta time-scale integral, the Cauchy nabla time-scales integrals and the Cauchy \( \alpha \)-diamond time scale integrals.

**Remark 7.** (i) Let \( a, b \in \mathbb{T} \). If \( f \in C_{rd}(\mathbb{T}, \mathbb{R}) \) is positive and

\[
\int_a^b f(x) \Delta x \geq (b - a)^{p-1}
\]

then

\[
\int_a^b f^p(x) \Delta x \left[ 1 - \frac{2}{\max\{p, q\}} \left( 1 - \frac{\int_a^b f^\frac{p}{q} \Delta x}{(b - a)^\frac{1}{q}(\int_a^b f^p \Delta x)^\frac{1}{q}} \right) \right]^p \geq \left[ \int_a^b f(x) \Delta x \right]^{p-1},
\]

where \( p > 1 \).

(ii) In the case of the Cauchy nabla time-scales integrals and the Cauchy \( \alpha \)-diamond time scale integrals similar inequalities can be stated as above.

**References**


Some integral inequalities


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