Another Look at Independent Domination in Graphs

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Abstract

In this paper we characterize the independent dominating sets in some graphs resulting from graph operations such as the corona, lexicographic product, and Cartesian product of graphs. Upper bounds or exact values of the independent domination numbers of these graphs are also determined.

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1 Introduction

Let \( G = (V(G), E(G)) \) be a simple graph. A subset \( S \) of \( V(G) \) is an independent set if for any two distinct elements \( x, y \in S \), \( xy \notin E(G) \). The independence number of \( G \), denoted by \( \beta(G) \), is the largest cardinality of an independent set in \( G \). An independent set \( S \) in \( G \) is called a maximum independent set if \( |S| = \beta(G) \). A subset \( D \) of \( V(G) \) is a dominating set in \( G \) if for every \( v \in V(G) \setminus D \), there exists \( x \in D \) such that \( xv \in E(G) \). If \( D \) is both independent and a dominating set, then it is an independent dominating set. The domination
number (resp. independent domination number) $\gamma(G)$ (resp. $\gamma_i(G)$) of $G$ is the smallest cardinality of a dominating (resp. independent dominating) set in $G$. A dominating (independent dominating) set $D$ in $G$ is called a minimum dominating (resp. minimum independent dominating) set if the cardinality of $D$ is equal to $\gamma(G)$ (resp. $\gamma_i(G)$). Other types of dominating sets and corresponding parameters are found in [8]. An upper bound of the Nordhaus-Gaddum-type product for the independent domination number was given in [5]. It was first improved by Cockayne et al. in 1991 (see [2]), and then by Cockayne et al. in 1995 (see [3]). The best possible upper bound was proved by Goddard and Henning in 2003 (see [7]), where they also gave an upper bound for the sum of the independent domination numbers of a graph and its complement. Other interesting results on independent domination of a graph are found in [4], [6], and [1].

2 Corona of Graphs

The corona $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex in the $i$th copy of $H$. We denote by $v + H^v$ a subgraph of $G \circ H$ obtained by joining the vertex $v \in V(G)$ to every vertex of the copy $H^v$ of $H$.

**Theorem 2.1** Let $G$ be a connected graph and $H$ any graph. Then $C \subseteq V(G \circ H)$ is an independent dominating set in $G \circ H$ if and only if $C \cap V(G)$ is an independent set in $G$ and $C \cap V(v + H^v)$ is an independent dominating set in $v + H^v$ for every $v \in V(G)$.

**Proof:** Suppose $C$ is an independent dominating set in $G \circ H$. Then $C \cap V(G)$ and $C \cap V(v + H^v)$ are independent sets in $G$ and $v + H^v$, respectively. Let $v \in V(G)$ and let $x \in V(v + H^v) \setminus C$. Since $x \in V(G \circ H) \setminus C$ and $C$ is a dominating set in $G \circ H$, $C \cap V(v + H^v) \neq \emptyset$. Hence, there exists $y \in C \cap V(v + H^v)$ such that $xy \in E(v + H^v)$. This shows that $C \cap V(v + H^v)$ is a dominating set in $v + H^v$.

Next, suppose that $C \cap V(G)$ is an independent set in $G$ and $C \cap V(v + H^v)$ is an independent dominating set in $v + H^v$ for every $v \in V(G)$. Let $C_1 = C \cap V(G)$ and $C_2 = C \setminus C_1$. For each $x \in C_2$, let $v_x \in V(G)$ such that $x \in V(H^{v_x})$. Then $C \cap V(v_x + H^{v_x})$ is an independent dominating set in $v_x + H^{v_x}$. Let $a$ and $b$ be distinct vertices in $C$. If $a, b \in C_1$, then $ab \notin E(G \circ H)$. Suppose $a \in C_1$ and $b \in C_2$. Then $b \in V(H^{v_b})$. Since $C \cap V(v_b + H^{v_b})$ is an independent set, $a \neq v_b$ (otherwise, $ab \notin E(v_b + H^{v_b})$). Thus $ab \notin E(G \circ H)$. Suppose that $a, b \in C_2$. If $v_a = v_b$, then $ab \notin E(G \circ H)$ since $C \cap V(v_b + H^{v_b})$ is an independent set. If $v_a \neq v_b$, then by definition of the corona $G \circ H$, $ab \notin E(G \circ H)$. This shows that $C$ is an independent set in $G \circ H$. 
Finally, let \( z \in V(G \circ H) \setminus C \). Since \( C \cap V(v_z + H^v) \) is a dominating set in \( v_z + H^v \), there exists \( x \in C \cap V(v_z + H^v) \) such that \( xz \in E(G \circ H) \). Therefore, \( C \) is a dominating set in \( G \circ H \). \( \square \)

**Theorem 2.2** Let \( G \) be a connected graph of order \( n \) and \( H \) any graph with \( \gamma_i(H) \neq 1 \). If \( C \subseteq V(G \circ H) \) is a minimum independent dominating set in \( G \circ H \) then \( C \cap V(G) \) is a maximum independent set in \( G \).

**Proof:** Let \( C_1 = C \cap V(G) \) and suppose it is not a maximum independent set in \( G \). Let \( M_1 \) be a maximum independent set in \( G \). Then \( k = |M_1| - |C_1| > 0 \). For each \( v \in V(G) \setminus C_1 \), let \( D_v = V(H^v) \cap C \). Then each \( D_v \) is an independent dominating set in \( v + H^v \) and \( C_2 = C \setminus C_1 = \cup \{D_v : v \in V(G) \setminus C_1 \} \). Let \( r = \min \{|D_v| : v \in V(G) \setminus C_1 \} \) and let \( v_0 \in V(G) \setminus C_1 \) such that \( r = |D_{v_0}| \). Let \( D \subseteq V(H) \) be such that \( \langle D \rangle \cong \langle D_{v_0} \rangle \). For each \( v \in V(G) \setminus M_1 \), let \( M_v \subseteq V(H^v) \) with \( \langle M_v \rangle \cong \langle D \rangle \). Let \( M_2 = \cup \{M_v : v \in V(G) \setminus M_1 \} \). By Theorem 2.1, \( C^* = M_1 \cup M_2 \) is an independent dominating set in \( G \circ H \). Now,

\[
|C| = |C_1| + |C_2| \geq |C_1| + \sum_{v \in V(G) \setminus C_1} |D_v| = |C_1| + (n - |C_1|)|D|.
\]

Since \( \gamma_i(H) \neq 1 \), it follows that \( |D| \geq 2 \). Thus

\[
|C| \geq |C_1| + (n - |C_1|)|D| > |C_1| + (n - |C_1|)|D| + k(1 - |D|).
\]

Since

\[
|C^*| = |M_1| + |M_2| = (|C_1| + k) + (n - |C_1| - k)|D|,
\]

it follows that \( |C| > |C^*| \). This is impossible because \( C \) is a minimum independent dominating set in \( G \circ H \). Therefore \( V(G) \cap C \) is a maximum independent set in \( G \). \( \square \)

**Corollary 2.3** Let \( G \) be a connected graph of order \( n \) and let \( H \) be any graph. Then \( \gamma_i(G \circ H) = \beta(G) + (n - \beta(G))\gamma_i(H) \).

**Proof:** Firstly, \( \gamma_i(H) \neq 1 \). Let \( C \) be a minimum independent dominating set in \( G \circ H \). Let \( C_1 = C \cap V(G) \) and \( C_2 = C \setminus C_1 \). Then \( C_1 \) is a maximum independent set in \( G \) by Theorem 2.2, hence \( |C_1| = \beta(G) \). Therefore

\[
\gamma_i(G \circ H) \geq \beta(G) + (n - \beta(G))\gamma_i(H).
\]

Next, let \( C_1 \) be a maximum independent set in \( G \) and \( D^* \) be a minimum independent dominating set in \( H \). For each \( v \in V(G) \setminus C_1 \), let \( D_v \subseteq V(H^v) \) be such that \( \langle D_v \rangle \cong \langle D^* \rangle \). Let \( C_2 = \cup \{D_v : v \in V(G) \setminus C_1 \} \). Then \( C^* = C_1 \cup C_2 \) is an independent dominating set in \( G \circ H \) by Theorem 2.1. Therefore

\[
\gamma_i(G \circ H) \leq |C^*| = \beta(G) + (n - \beta(G))\gamma_i(H).
\]
This establishes the required equality.

Suppose now that \( \gamma_i(H) = 1 \). For each \( v \in V(G) \), let \( D^v = \{ x_v \} \) be a (independent) dominating set in the copy \( H^v \) of \( H \). Then, clearly, \( C = \bigcup \{ D^v : v \in V(G) \} \) is a minimum independent dominating set in \( G \circ H \) by Theorem 2.1. Therefore, \( \gamma_i(G \circ H) = |C| = n \). Since \( \gamma_i(H) = 1 \), \( \beta(G) + (n - \beta(G))\gamma_i(H) = n \). \( \square \)

## 3 Lexicographic Product of Graphs

The lexicographic product \( G[H] \) of two graphs \( G \) and \( H \) is the graph with \( V(G[H]) = V(G) \times V(H) \) and \( (u, u')(v, v') \in E(G[H]) \) if and only if either \( uv \in E(G) \) or \( u = v \) and \( u'v' \in E(H) \).

Observe that any non-empty subset \( C \) of \( V(G[H]) = V(G) \times V(H) \) (in fact, any set of ordered pairs) can be expressed or written as \( C = \bigcup_{x \in S} \{ x \} \times T_x \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \). Henceforth, we shall use this form to denote any non-empty subset \( C \) of \( V(G) \times V(H) \).

**Theorem 3.1** Let \( G \) and \( H \) be connected graphs. A subset \( C = \bigcup_{x \in S} \{ x \} \times T_x \) of \( V(G[H]) \), is an independent dominating set in \( G[H] \) if and only if \( S \) is an independent dominating set in \( G \) and \( T_x \) is an independent dominating set in \( H \) for every \( x \in S \).

**Proof:** Suppose \( C = \bigcup_{x \in S} \{ x \} \times T_x \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \) is an independent dominating set in \( G[H] \). Let \( u \in V(G) \setminus S \) and pick \( b \in V(H) \). Since \( C \) is a dominating set in \( G[H] \), there exists \( (y, c) \in C \) such that \( (y, c)(u, b) \in E(G[H]) \). This implies that \( y \in S \) and \( u \in N_G(y) \). This shows that \( S \) is a dominating set in \( G \). Let \( x, y \in S \), where \( x \neq y \). Pick \( c \in T_x \) and \( d \in T_y \). Then \( (x, c), (y, d) \in C \) and \( (x, c) \neq (y, d) \). Since \( C \) is an independent set, \( (x, c)(y, d) \notin E(G[H]) \). It follows that \( xy \notin E(G) \). Thus, \( S \) is an independent set in \( G \).

Next, let \( x \in S \) and let \( a \in V(H) \setminus T_x \). Then \( (x, a) \notin C \). Again, since \( C \) is a dominating set, there exists \( (y, b) \in C \) such that \( (x, a)(y, b) \in E(G[H]) \). Since \( S \) is an independent set, \( xy \notin E(G) \). Therefore, \( x = y \) and \( ab \in E(H) \), where \( b \in T_x = T_y \). This shows that \( T_x \) is a dominating set in \( H \). Now, suppose that \( p \) and \( q \) are distinct vertices in \( T_x \). If \( pq \in E(H) \), then \( (x, p)(x, q) \in E(G[H]) \). Since \( (x, p), (x, q) \in C \), it follows that \( C \) is not an independent set, contrary to our assumption. Therefore, \( pq \notin E(H) \). This implies that \( T_x \) is an independent set in \( H \).

For the converse, suppose that \( S \) is an independent dominating set in \( G \) and \( T_x \) is an independent dominating set in \( H \) for every \( x \in S \). Let \( (u, v) \in V(G[H]) \setminus C \). Consider the following cases:
Case 1. Suppose $u \in S$. Then $v \notin T_u$. Since $T_u$ is a dominating set in $H$, there exists $w \in T_u$ such that $vw \in E(H)$. Thus, $(u, w) \in C$ and $(u, v)(u, w) \in E(G[H])$. This shows that $C$ is a dominating set in $G[H]$.

Case 2. Suppose $u \notin S$. Since $S$ is a dominating set in $G$, there exists $z \in S$ such that $uz \in E(G)$. Choose $b \in T_z$. Then $(z, b) \in C$ and $(u, v)(z, b) \in E(G[H])$.

Therefore, $C$ is a dominating set in $G[H]$.

Finally, let $(x, a), (y, b) \in C$, where $(x, a) \neq (y, b)$. Suppose $x = y$. Then $a, b \in T_x$ and $a \neq b$. Since $T_x$ is an independent set in $H$, it follows that $ab \notin E(H)$. Therefore $(x, a)(y, b) \notin E(G[H])$. If $x \neq y$, then $x$ and $y$ are distinct vertices in $S$. Since $S$ is an independent set in $G$, $xy \notin E(G)$. Thus $(x, a)(y, b) \notin E(G[H])$. Accordingly, $C$ is an independent set in $G[H]$. □

The following result is a direct consequence of Theorem 3.1.

**Corollary 3.2** Let $G$ and $H$ be connected graphs. Then

$$\gamma_i(G[H]) = \gamma_i(G)\gamma_i(H).$$

**Proof**: Let $C = \bigcup_{x \in S}(\{x\} \times T_x)$ be a minimum independent dominating set. By Theorem 3.1, $S$ is an independent dominating set in $G$ and $T_x$ is an independent dominating set in $H$ for every $x \in S$. Thus

$$\gamma_i(G[H]) = |C| \geq |S|\gamma_i(H) = \gamma_i(G)\gamma_i(H).$$

Now let $S$ and $D$ be minimum independent dominating sets in $G$ and $H$, respectively. For each $x \in S$, set $T_x = D$. Then $C = \bigcup_{x \in S}(\{x\} \times T_x)$ is an independent dominating set in $G[H]$, by Theorem 3.1. Therefore,

$$\gamma_i(G[H]) \leq |C| = |S||D| = \gamma_i(G)\gamma_i(H).$$

Accordingly, $\gamma_i(G[H]) = \gamma_i(G)\gamma_i(H)$. □

Since $\gamma_i(K_n) = 1$, the following result is immediate from Corollary 3.2.

**Corollary 3.3** Let $G$ be a connected graph and $K_n$ the complete graph of order $n \geq 1$. Then $\gamma_i(G[K_n]) = \gamma_i(G)$.

### 4 Cartesian Product of Graphs

The *cartesian product* $G \times H$ of two graphs $G$ and $H$ is the graph with $V(G \times H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \times H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$. 
Let $C$ be a subset of $V(G \times H)$. Then the $G$-projection $C_G$ and $H$-projection $C_H$ of $C$ are the following sets:

$$C_G = \{ x \in V(G) : (x, a) \in C \text{ for some } a \in V(H) \}$$

$$C_H = \{ b \in V(H) : (y, b) \in C \text{ for some } y \in V(G) \}$$

Further, if $x \in C_G$, then $T_x = \{ a \in C_H : (x, a) \in C \}$ and if $a \in C_H$, then $D_a = \{ x \in C_G : (x, a) \in C \}$.

**Theorem 4.1** Let $G$ and $H$ be non-trivial connected graphs. Then a subset $C$ of $V(G \times H)$ is an independent dominating set in $G \times H$ if and only if either

(i) $C_G = V(G)$ such that

(a) for each $x \in V(G)$, $T_x$ is an independent set in $H$, and for every $p \in V(H) \setminus N_H[T_x]$, there exists $y \in N_G(x)$ with $p \in T_y$, and

(b) $T_x \cap T_y = \emptyset$ whenever $x, y \in V(G)$ with $xy \in E(G)$, or

(ii) $C_H = V(H)$ such that

(c) for each $p \in V(H)$, $D_p$ is an independent set in $G$, and for every $x \in V(G) \setminus N_G[D_p]$, there exists $q \in N_H(p)$ with $x \in D_q$, and

(d) $D_p \cap D_q = \emptyset$ whenever $p, q \in V(H)$ with $pq \in E(H)$.

**Proof:** Suppose $C$ is a dominating set in $G \times H$. Suppose further that $C_G \neq V(G)$ and $C_H \neq V(H)$. Let $x \in V(G) \setminus C_G$ and $a \in V(H) \setminus C_H$. Then $(x, a) \notin C$. Since $C$ is a dominating set in $G \times H$, there exists $(u, c) \in C$ such that $(u, c)(x, a) \in E(G \times H)$. This implies that $u = x$ and $ca \in E(H)$ or $c = a$ and $ux \in E(G)$, hence $x \in C_G$ or $a \in C_H$, contradicting our assumption. Thus, $C_G = V(G)$ or $C_H = V(H)$. Suppose that $C_G = V(G)$. Let $x \in V(G)$ and let $t_1, t_2 \in T_x$ with $t_1 \neq t_2$ Then $(x, t_1), (x, t_2) \in C$ and $(x, t_1) \neq (x, t_2)$. Since $C$ is independent, $(x, t_1)(x, t_2) \notin E(G \times H)$. This implies that $t_1 t_2 \notin E(H)$, showing that $T_x$ is an independent set in $H$. Now, if $p \in V(H) \setminus N_H[T_x]$, then $p \notin T_x$ and $pq \notin E(H)$ for all $q \in T_x$. Since $C$ is a dominating set, there exists $(y, s) \in C$ such that $(x, p)(y, s) \in E(G \times H)$. This implies that $y \in N_G(x)$ and $p \in T_y$. This shows that (a) holds. Finally, let $x, y \in V(G)$ with $xy \in E(G)$. Suppose $t \in T_x \cap T_y$. Then $(x, t), (y, t) \in C$ and $(x, t)(y, t) \in E(G \times H)$. This implies that $C$ is not an independent set, contrary to our assumption. Thus $T_x \cap T_y = \emptyset$, showing that condition (b) also holds.

If $C_H = V(H)$, then it can be shown in a similar manner that (c) and (d) both hold.

For the converse, suppose first that (i) holds. Let $(x, p) \in V(G \times H) \setminus C$. Then $p \notin T_x$. Consider the following cases:
Case1: Suppose that \( p \in N_G(T_x) \). Then there exists \( q \in T_x \) such that \( pq \in E(H) \). It follows that \((x, q) \in C \) and \((x, p)(x, q) \in E(G \times H) \).

Case2: Suppose that \( p \notin N_G(T_x) \). By (a), there exists \( y \in N_G(x) \) such that \( p \in T_y \). It follows that \((y, p) \in C \) and \((x, p)(y, p) \in E(G \times H) \).

Accordingly, \( C \) is a dominating set in \( G \times H \).

Next, let \((v, t), (w, s) \in C \) with \((v, t) = (w, s) \). If \( v = w \), then \( s, t \in T_v = T_w \). Since \( T_v \) is an independent set, \( st \notin E(H) \). It follows that \((v, t)(w, s) \notin E(G \times H) \). Suppose that \( v \neq w \). If \( vw \notin E(G) \), then \((v, t)(w, s) \notin E(G \times H) \). If \( vw \in E(G) \), then \( T_v \cap T_w = \emptyset \) by (b). It follows that \( t \neq s \) and \((v, t)(w, s) \notin E(G \times H) \). Therefore, \( C \) is an independent set in \( G \times H \).

Similarly, \( C \) is an independent dominating set in \( G \times H \) if property (ii) holds. \( \square \)

**Corollary 4.2** Let \( G \) and \( H \) be non-trivial connected graphs. Then
\[
\gamma_i(G \times H) \leq \min\{\beta(G)|V(H)|, \beta(H)|V(G)|\}.
\]

**Proof:** Let \( C \) be an independent dominating set in \( G \times H \). If \( C_G = V(G) \), then by Theorem 4.1(i), \( T_x \) is an independent set in \( H \) for each \( x \in V(G) \). Since \( C = \bigcup_{x \in V(G)} \{x\} \times T_x \), it follows that \( |C| = \sum_{x \in V(G)} |T_x| \leq |V(G)|\beta(H) \).

If \( C_H = V(H) \), then by Theorem 4.1(ii), \( D_p \) is an independent set in \( G \) for each \( p \in V(H) \). Since \( C = \bigcup_{p \in V(H)} (D_p \times \{p\}) \), \( |C| = \sum_{p \in V(H)} |D_p| \leq |V(H)|\beta(G) \).

The desired result now follows. \( \square \)

Observe that the upper bound in Corollary 4.2 is attained for \( G = K_3 \) and \( H = K_3 \), i.e., \( \gamma_i(K_3 \times K_3) = 3 = \beta(K_3)|V(K_3)| \).

**Corollary 4.3** Let \( G \) be a non-trivial connected graph and \( n \geq 2 \).

(i) If \( |V(G)| \leq n \), then \( \gamma_i(G \times K_n) = |V(G)| \).

(ii) Let \( |V(G)| > n \). Then \( \gamma_i(G \times K_n) = n \) if and only if there exists \( S \subseteq V(G) \) such that \( |S| = n \) and \( V(G) \setminus S \subseteq N_G(x) \) for each \( x \in S \).

**Proof:** (i) Suppose \( |V(G)| \leq n \) and let \( C \) be a minimum independent dominating set in \( G \times H \). By Theorem 4.1, \( C_G = V(G) \) or \( C_H = V(K_n) \). If \( C_G = V(G) \), then \( |V(G)| = |C_G| \leq |C| = \gamma_i(G \times K_n) \). If \( C_H = V(K_n) \), then \( |V(G)| \leq n = |C_H| \leq |C| = \gamma_i(G \times K_n) \). It follows that \( |V(G)| \leq \gamma_i(G \times K_n) \).

Now let \( V(G) = \{x_1, x_2, \ldots, x_m\} \) and \( V(K_n) = \{a_1, a_2, \ldots, a_n\} \). Let \( C^* = \{(x_1, a_1), (x_2, a_2), \ldots, (x_m, a_m)\} = \bigcup_{i=1}^m \{x_i\} \times T_{x_i} \) , where \( T_{x_i} = \{a_i\} \) for \( i = 1, 2, \ldots, m \). By Theorem 4.1(i), \( C^* \) is an independent dominating set in \( G \times K_n \). It follows that \( \gamma_i(G \times K_n) = |C| \leq |C^*| = m = |V(G)| \). Therefore \( \gamma_i(G \times K_n) = |V(G)| \).
(ii) Suppose $|V(G)| > n$ and suppose $\gamma_i(G \times K_n) = n$. Let $C$ be a minimum independent dominating set of $G \times K_n$. Then $C_G \neq V(G)$. It follows that $C_H = V(K_n)$ by Theorem 4.1. Since $|C| = n$, $D_p$ is a singleton for each $p \in V(K_n)$. Let $S = \cup_{p \in V(K_n)} D_p$. Then $|S| = n$. Let $z \in V(G) \setminus S$ and let $x \in S$. Let $q \in V(K_n)$ such that $D_q = \{x\}$. Since $(z, t) \notin C$ for all $t \in V(K_n)$ (in particular, $(z, a) \notin C$) and $C$ is a dominating set, $(z, a) \in E(G \times V(K_n))$. This implies that $z \in N_G(x)$. This shows that $V(G) \setminus S \subseteq N_G(x)$ for each $x \in S$.

For the converse, suppose that such a subset $S$ of $V(G)$ exists. Let $S = \{x_1, x_2, \ldots, x_n\}$ and $V(K_n) = \{p_1, p_2, \ldots, p_n\}$. Let $D_{p_i} = \{x_i\}$ for each $i = 1, 2, \ldots, n$ and set $C = \{(x_1, a_1), (x_2, a_2), \ldots, (x_n, a_n)\} = \cup_{k=1}^{n} (D_{p_k} \times \{p_k\})$. Then $C$ satisfies (ii) of Theorem 4.1; hence, $C$ is an independent dominating set in $G \times K_n$). Therefore $\gamma_i(G \times K_n) \leq |C| = n$. Since $\gamma_i(G \times K_n) \geq n$, it follows that $\gamma_i(G \times K_n) = n$. □

References


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