Secure Connected Domination in a Graph

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Abstract

Let $G = (V(G), E(G))$ be a connected simple graph. A connected dominating set $S$ of $V(G)$ is a secure connected dominating set of $G$ if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\} \cup \{u\})$ is a connected dominating set of $G$. The minimum cardinality of a secure connected dominating set of $G$, denoted by $\gamma_{sc}(G)$, is called the secure connected domination number of $G$. We characterized secure connected dominating set in terms of the concept of external private neighborhood of a vertex. Also, we give necessary and sufficient conditions for connected graphs to have secure connected domination number equal to 1 or 2. The secure connected domination numbers of graphs resulting from some binary operations are also investigated.

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1 Introduction

Let \( G = (V(G), E(G)) \) be a simple graph with vertex set \( V(G) \) of finite order and edge set \( E(G) \). The neighborhood of \( v \) is the set \( N_G(v) = N(v) = \{ u \in V(G) : uv \in E(G) \} \). If \( X \subseteq V(G) \), then the open neighborhood of \( X \) is the set \( N_G(X) = N(X) = \bigcup \{ N_G(v) : v \in X \} \). The closed neighborhood of \( X \) is \( N_G[X] = N[X] = X \cup N(X) \). A vertex \( w \in V(G) \setminus X \) is an \( X \)-external private neighbor if \( N_G(w) \cap X = \{ v \} \). The set of all external private neighbors of \( v \in X \) is denoted by \( \text{epn}(v, X) \).

A subset \( S \) of \( V(G) \) is a dominating set (DS) in \( G \) if for every \( u \in V(G) \setminus S \), there exists \( v \in S \) such that \( uv \in E(G) \), i.e., \( N[S] = V(G) \). The domination number of \( G \) is the minimum cardinality of a dominating set in \( G \) and is denoted by \( \gamma(G) \). A set \( S \) is said to be a secure dominating set in \( G \) if for every \( u \in V(G) \setminus S \) there exists \( v \in S \) such that \( uv \in E(G) \) and \( (S \setminus \{ v \}) \cup \{ u \} \) is a dominating set. The minimum cardinality of a secure dominating set in \( G \) is called the secure domination number of \( G \) and is denoted by \( \gamma_s(G) \). This variant of domination was introduced in [5] and several studies is found in [1, 2, 4] and elsewhere.

Another variant of domination was introduced by Sampathkumar and Walikar in [8] which they called connected domination. A dominating set \( S \) is said to be connected dominating set (CDS), if the induced subgraph \( \langle S \rangle \) is connected. Let \( S \) be a connected dominating set in \( G \). A vertex \( u \in S \) is said to \( S \)-defend \( u \), where \( u \in V(G) \setminus S \), if \( uv \in E(G) \) and \( (S \setminus \{ v \}) \cup \{ u \} \) is a connected dominating set in \( G \). \( S \) is a secure connected dominating set (SCDS) in \( G \) if for each \( u \in V(G) \setminus S \), there exists \( v \in S \) such that \( v \) \( S \)-defends \( u \). The secure connected domination number \( \gamma_{sc}(G) \) of \( G \) is the smallest cardinality of a secure connected dominating set in \( G \). A dominating set (resp. secure connected dominating set) \( S \) in \( G \) with \( |S| = \gamma(G) \) (resp. \( |S| = \gamma_{sc}(G) \)) is called a \( \gamma \)-set (resp. \( \gamma_{sc} \)-set).

In this paper, we initiate a study of secure connected domination in graphs. Any undefined terms maybe found in [3] or [6].

2 Results and Discussion

Remark 2.1 Let \( G \) be a connected graph of order \( n \). Then \( 1 \leq \gamma_{sc}(G) \leq n \).

Theorem 2.2 Let \( S \) be a connected dominating set of \( G \), \( v \in S \) and \( u \in V(G) \setminus S \) with \( uv \in E(G) \). \( v \) \( S \)-defends \( u \) if and only if \( \text{epn}(v, S) \subseteq N_G[u] \) and \( V(C) \cap N_G(u) \neq \emptyset \), for every component \( C \) of \( \langle S \setminus \{ v \} \rangle \).
Secure connected domination in a graph

Proof: Suppose $v$ S-defends $u$. Let $w \in \text{epn}(v, S)$. Then $w \notin S$ and $N_G(w) \cap S = \{v\}$. If $w = u$, then $w \in N_G(u)$. If $w \neq u$, then $w \notin T$, where $T = (S \setminus \{v\}) \cup \{u\}$. Since $T$ is a dominating set in $G$ and $N_G(w) \cap S = \{v\}$, $w \in N_G(u)$. Hence, $\text{epn}(v, S) \subseteq N_G[u]$. Suppose that $V(C) \cap N_G(u) = \emptyset$ for some component $C$ of $\langle S \setminus \{v\} \rangle$. Then there is no path joining any $a \in V(C)$ and $u$; hence $T$ is not connected which is contrary to the assumption that $v$ S-defends $u$.

Now we suppose that $\text{epn}(v, S) \subseteq N_G[u]$ and $V(C) \cap N_G(u) \neq \emptyset$, for every component $C$ of $\langle S \setminus \{v\} \rangle$. Let $w \in V(G) \setminus T$, where $T = (S \setminus \{v\}) \cup \{u\}$. If $w = v$, then $uw \in E(G)$. If $w \neq v$, then $w \notin S$. If $w \notin \text{epn}(v, S)$, then $uw \in E(G)$. If $w \notin \text{epn}(v, S)$, then $N_G(w) \cap S \neq \{v\}$. Hence, there exists $t \in T \setminus \{v\} \subset T$ such that $tw \in E(G)$. Thus, $T$ is a dominating set in $G$.

It remains to show that $\langle T \rangle$ is connected. Let $a, b \in T$, where $a \neq b$ and $ab \notin E(G)$. Consider the following cases:

Case 1: $a = u$ and $b \in V(D_1)$, where $D_1$ is a component of $\langle S \setminus \{v\} \rangle$.

Let $z \in V(D_1) \cap N_G(a)$. Then $az \in E(G)$ and $z \in V(D_1)$. Since $D_1$ is connected, there is a path, say $P(z, b)$, joining $z$ and $b$. Now, the path $P(z, b)$ together with the edge $az$ forms a path $P(a, b)$ joining $a$ and $b$.

Case 2: $a$ and $b$ are in the same component $D$ of $\langle S \setminus \{v\} \rangle$.

Since $D$ is connected, there is a path joining $a$ and $b$.

Case 3: $a \in V(D_1)$ and $b \in V(D_2)$, where $D_1$ and $D_2$ are different components of $\langle S \setminus \{v\} \rangle$.

Let $w \in V(D_1) \cap N_G(u)$ and $x \in V(D_2) \cap N_G(u)$. Since $D_1$ is connected, there is a path $P(a, w)$ joining $a$ and $w$. Similarly, there is also a path $P(x, b)$ joining $x$ and $b$. The paths $P(a, w)$ and $P(x, b)$, together with the edges $uw$ and $ux$ form a path $P(a, b)$ from $a$ to $b$.

Thus, in either case, $\langle T \rangle$ is connected. \qed

Corollary 2.3 Let $S$ be a connected dominating set in $G$. Then $S$ is a secure connected dominating set in $G$ if and only if for every $u \in V(G) \setminus S$, $\exists v \in S \cap N_G(u)$ such that

(i) $\text{epn}(v, S) \subseteq N_G[u]$, and

(ii) $V(C) \cap N_G(u) \neq \emptyset$, for every component $C$ of $\langle S \setminus \{v\} \rangle$.

Proof: Suppose $S$ is an SCDS in $G$. Then for every $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $\langle S \setminus \{v\} \rangle \cup \{u\}$ is a CDS in $G$. This means that $v$ S-defends $u$. By Theorem 2.2, (i) and (ii) hold. The converse follows immediately. \qed

A leaf $u$ of a graph $G$ is a vertex of degree one and the support vertex of a leaf $u$ is the unique vertex $v$ such that $uv \in E(G)$. We denote by $L(G)$ and $S(G)$ be the set of leaves and support vertices of $G$, respectively.
Theorem 2.4 Let $G$ be a connected graph of order $n \geq 3$ and let $X$ be a secure connected dominating set of $G$. Then

(i) $L(G) \subseteq X$ and $S(G) \subseteq X$

(ii) No vertex in $L(G) \cup S(G)$ is an $X$-defender.

Proof: Suppose that $x \in L(G)$. Let $y \in S(G)$ such that $xy \in E(G)$. Let $S^* = (X \setminus \{y\}) \cup \{x\}$. Since $\langle S^* \rangle$ is not connected, $x \in X$. Suppose $x \in S(G)$. Let $z \in L(G) \cap N_G(x)$. Then $z \in X$. Since $\langle X \rangle$ is connected, $x \in X$.

Let $v \in L(G) \cup S(G)$ and $u \in V(G) \setminus X$. Now $v \in L(G) \cup S(G) \Rightarrow v \in L(G)$ or $v \in S(G)$. If $v \in L(G)$, then $uv \notin E(G)$. If $v \in S(G)$, then there exists $z \in V(G)$ such that $N(z) = \{v\}$. By (i), $z \in X$. Thus $\langle X \setminus \{v\} \rangle \cup \{u\}$ is not connected. Thus, in either case, $v$ does not $X$-defend $u$. \hfill \Box

Theorem 2.5 Let $n$ be a positive integer. Then

(i) $\gamma_{sc}(K_n) = 1$ for all $n \geq 2$

(ii) $\gamma_{sc}(P_n) = \begin{cases} 1, & \text{if } n = 2 \\ n, & \text{if } n \geq 3 \end{cases}$

(iii) $\gamma_{sc}(C_n) = \begin{cases} 1, & \text{if } n = 3 \\ n-1, & \text{if } n > 3 \end{cases}$

Proof: First suppose that $G = K_n$. Then $S = \{v\} \subseteq V(G)$ is a connected dominating set of $G$. Now $\forall x \in V(G)$, $(S \setminus \{v\}) \cup \{x\} = \{x\}$ which is a CDS of $G$. Thus $S = \{v\}$ is an SCDS of $G$. Therefore, $\gamma_{sc}(G) \leq 1$. By Remark 2.1, it follows that $\gamma_{sc}(G) = 1$. Next, let $P_n$ be a path of order $n \geq 2$. For $n = 2$, $P_2 \cong K_2$ and $\gamma_{sc}(P_2) = 1$. Suppose $n \geq 3$. Let $S$ be a $\gamma_{sc}$-set of $P_n = [v_1, v_2, \ldots, v_n]$. Then $v_1, v_n \in S$ by Theorem 2.4. Since $\langle S \rangle$ is connected, $S = V(P_n)$. Thus $\gamma_{sc}(P_n) = |S| = n$. Finally, let $C_n$ be a cycle of order $n \geq 3$. Suppose $n = 3$, then $C_3 \cong K_3$ and $\gamma_{sc}(C_3) = 1$. For $n > 3$, let $C_n = [v_1, v_2, \ldots, v_n, v_1]$ and let $S^* = \{v_1, v_2, \ldots, v_{n-1}\}$. Clearly, $S^*$ is an SCDS in $C_n$. Let $S$ be a $\gamma_{sc}$-set of $C_n$. Since $S^*$ is an SCDS in $C_n$, $|S| \leq |S^*| = n-1$. Suppose $|S| = k < n-1$. Since $\langle S \rangle$ is connected, $\langle V(C_n) \setminus S \rangle$ is connected, where $|V(C_n) \setminus S| = m \geq 2$. Moreover, since $S$ is a dominating set, $m = 2$. Let $V(C_n) \setminus S = \{v_i, v_{i+1}\}$, $i = 1, 2, \ldots, n-1$. Note that $\langle S \setminus \{v_{i-1}\} \rangle \cup \{v_i\} = \{v_1, v_2, \ldots, v_{i-2}, v_i, v_{i+1}, \ldots, v_n\}$ is not connected, which is contrary to the assumption that $|S| = k < n-1$. This shows that $|S| = k = n-1$. Consequently, $\gamma_{sc}(C_n) = n-1$. \hfill \Box
Theorem 2.6 Let $G$ be a connected graph of order $n$. Then $\gamma_{sc}(G) = 1$ if and only if $G = K_n$.

Proof: Suppose $\gamma_{sc}(G) = 1$ and let $S = \{v\}$ be an SCDS of $G$. Suppose $G \neq K_n$, then there exists $x, y \in V(K_n)$ such that $d(x, y) = 2$. Then $(S \setminus \{v\}) \cup \{x\} = \{x\}$, which is not a dominating set of $G$, since $xy \notin E(K_n)$. Therefore, $G = K_n$. The converse is true by Theorem 2.5. \hfill \Box

The following Theorem shows that it is not possible to find a non-complete connected graph $G$ such that $\gamma(G) = \gamma_{sc}(G)$.

Theorem 2.7 Let $G$ be a non-complete connected graph and let $S$ be a secure connected dominating set in $G$. Then the set $S \setminus \{v\}$ is a dominating set for every $v \in S$. In particular, $1 + \gamma(G) \leq \gamma_{sc}(G)$.

Proof: Let $v \in S$ and $T = S \setminus \{v\}$. We will show that $T$ is a dominating set in $G$. Let $w \in V(G) \setminus T$. If $w = v$, then $\exists u \in S$ such that $uw \in E(G)$ because $\langle S \rangle$ is connected. If $w \neq v$, then $w \notin S$. Since $S$ is a secure dominating set in $G$, there exists $x \in S \cap N_G(w)$ such that $x$ $S$-defends $w$. If $x \neq v$, then $x \in T$. If $x = v$, then $(S \setminus \{v\}) \cup \{w\}$ is a CDS. This implies that $\exists u \in T$ such that $uw \in E(G)$. Therefore, in any case, $T = S \setminus \{v\}$ is a DS. Thus, $1 + \gamma(G) \leq |S|$. If, in particular, $S$ is a $\gamma_{sc}$-set, then $1 + \gamma(G) \leq \gamma_{sc}(G)$. \hfill \Box

Theorem 2.8 Let $G$ be a connected graph of order $n \geq 4$. Then $\gamma_{sc}(G) = 2$ if and only if there exists a non-complete graph $H$ such that $G = K_2 + H$.

Proof: Suppose $G = K_2 + H$, for some non-complete graph $H$. Then $G$ is not complete and by Theorem 2.6, it follows that $\gamma_{sc}(G) \geq 2$. Let $S = \{x, y\}$, where $x, y \in V(K_2)$. Then $S$ is a connected dominating set in $G$. Let $z \in V(G) \setminus S$. Then $z \in V(H), xz \in E(G)$ and $(S \setminus \{x\}) \cup \{z\} = \{y, z\}$ a connected dominating set of $G$. Thus $S$ is an SCDS in $G$. Therefore, $\gamma_{sc}(G) = 2$.

For the converse, suppose that $\gamma_{sc}(G) = 2$. Let $S = \{x, y\}$ be a $\gamma_{sc}$-set of $G$. Let $z \in V(G) \setminus S$. Suppose $xz \notin E(G)$. Since $S$ is an SCDS of $G$, $yz \in E(G)$ and $S^* = \{x, z\}$ is a connected dominating set of $G$. This is not possible because $xz \notin E(G)$. Therefore, $xz \in E(G)$. Similarly, $yz \in E(G)$. Let $H = \langle V(G) \setminus S \rangle$ and let $K_2 = \langle S \rangle$. Then $G = K_2 + H$. Moreover, since $G$ is not complete, $H$ is non-complete. \hfill \Box

Corollary 2.9 Let $G$ be a non-complete connected graph and $n \geq 2$. Then $\gamma_{sc}(G + K_n) = 2$.

Corollary 2.10 Let $G$ be a non-complete connected graph. Then $\gamma_{sc}(K_1 + G) = 2$ if and only if one of the following is true:

(i) $\gamma(G) = 1$
(ii) $\gamma_{sc}(G) = 2$

**Proof:** Suppose $\gamma_{sc}(K_1 + G) = 2$ and let $V(K_1) = \{a\}$. Let $S = \{x, b\}$ be an SCDS in $K_1 + G$, where $b \in V(G)$. Consider the following cases:

**Case 1.** Suppose $x = a$.

If $\{b\}$ were not a dominating set of $G$, then $\exists y \in V(G) \setminus \{b\}$ such that $by \notin E(G)$. Since $S$ is an SCDS of $K_1 + G$, $S \setminus \{x\} \cup \{y\} = \{y, b\}$ is a connected dominating set of $K_1 + G$. This, however, is impossible to happen since $by \notin E(G)$. Thus, $\{b\}$ is a dominating set of $G$. This shows that $\gamma(G) = 1$.

**Case 2.** Suppose $x \neq a$.

Then $x \in G$. Hence, $S$ is an SCDS in $G$. Since $K_1 + G$ is not complete, it follows that $\gamma_{sc}(G) = |S| = 2$.

For the converse, suppose first that $\gamma(G) = 1$, say $\{b\}$ is a dominating set of $G$. Let $H = \langle V(G) \setminus \{b\} \rangle$. Then $H$ is non-complete and $K_1 + G \approx \langle \{a, b\} \rangle + H$. By Theorem 2.8, $\gamma_{sc}(K_1 + G) = 2$. Suppose now that $\gamma_{sc}(G) = 2$. Then by Theorem 2.8, $G = K_2 + H^*$, where $H^*$ is a non-complete graph. Let $H = \langle a \rangle + H^*$. Then $H$ is non-complete and $K_1 + G \approx K_2 + H$. Therefore, by Theorem 2.8, $\gamma_{sc}(K_1 + G) = 2$.

**Theorem 2.11** Let $G$ be a non-complete connected graph and $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a SCDS of $K_1 + G$ iff one of the following holds:

(i) $S \subseteq V(G)$ and $S$ is a SCDS of $G$.

(ii) $v \in S$ and $S \setminus \{v\}$ is a dominating set in $G$.

**Proof:** Suppose $S \subseteq V(K_1 + G)$ is an SCDS in $K_1 + G$. If $S \subseteq V(G)$, then $S$ is an SCDS of $G$. Suppose $v \in S$ and let $x \in V(G) \setminus (S \setminus \{v\})$. Suppose $xy \in E(K_1 + G)$ for all $y \in V(G) \setminus (S \setminus \{v\})$. Then $(S \setminus \{v\}) \cup \{x\}$ is not connected, contrary to our assumption that $S$ is an SCDS in $K_1 + G$. Thus, $S \setminus \{v\}$ is a dominating set in $G$.

For the converse suppose first that $S$ is an SCDS in $G$. Then $S$ is an SCDS in $K_1 + G$. Next suppose that $v \in S$ and $S \setminus \{v\}$ is a dominating set in $G$. Then $S$ is a connected dominating set in $K_1 + G$. Let $z \in V(K_1 + G) \setminus S$. Then $z \in V(G) \setminus (S \setminus \{v\})$. Since $S \setminus \{v\}$ is a DS in $G$, there exists $x \in S \setminus \{v\}$ such that $xz \in E(G)$ and $(S \setminus \{x\}) \cup \{z\}$ is a CDS in $K_1 + G$. Thus, $S$ is an SCDS in $K_1 + G$.

**Corollary 2.12** Let $G$ be a non-complete connected graph. Then

$$\gamma_{sc}(K_1 + G) = \min\{\gamma_{sc}(G), \gamma(G) + 1\}.$$
Proof: Let \( S_1 \) and \( S_2 \) be, respectively, \( \gamma_{sc} \)-set and \( \gamma \)-set of \( G \). By Theorem 2.11, \( S_1 \) and \( S_2 \cup V(K_1) \) are SCDS in \( K_1 + G \). Thus, \( \gamma_{sc}(K_1 + G) \leq \min\{|S_1|, |S_2 \cup V(K_1)|\} = \min\{\gamma_{sc}(G), \gamma(G) + 1\} \). Let \( r = \min\{\gamma_{sc}(G), \gamma(G) + 1\} \). Let \( S \) be a \( \gamma_{sc} \)-set of \( K_1 + G \). By Theorem 2.11, \( \gamma_{sc}(K_1 + G) = |S| \geq r \). This proves the desired equality. \( \square \)

From this result, we can readily solve for the secure connected domination number of the graphs: \( F_n \) and \( W_n \).

Remark 2.13 Let \( F_n \), and \( W_n \) be the fan and wheel of order \( n + 1 \), respectively. Then

\[
(i) \quad \gamma_{sc}(F_n) = \begin{cases} 1, & \text{if } n = 1 \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \geq 2 \end{cases}
\]

\[
(ii) \quad \gamma_{sc}(W_n) = \begin{cases} 1, & \text{if } n = 3 \\ \lceil \frac{n}{3} \rceil + 1, & \text{if } n \geq 4 \end{cases}
\]

Remark 2.14 Corollary 2.10 also follows from Corollary 2.12.

Let \( G \) and \( H \) be non-complete connected graphs. Then \( G + H \) is non-complete. Thus \( \gamma_{sc}(G + H) \geq 2 \). Now, choose \( a, b \in V(G) \) and \( x, y \in V(H) \). Then \( S = \{a, b, x, y\} \) is a SCDS in \( G + H \). It follows that \( \gamma_{sc}(G + H) \leq 4 \). This result is stated formally in the following remark.

Remark 2.15 Let \( G \) and \( H \) be non-complete connected graphs. Then \( 2 \leq \gamma_{sc}(G + H) \leq 4 \).

Theorem 2.16 Let \( G \) and \( H \) be non-complete connected graphs. Then \( \gamma_{sc}(G + H) = 2 \) if and only if one of the following holds:

\[(i) \quad \gamma_{sc}(G) = 2.\]

\[(ii) \quad \gamma_{sc}(H) = 2.\]

\[(iii) \quad \gamma(G) = 1 \text{ and } \gamma(H) = 1.\]

Proof: Suppose \( \gamma_{sc}(G + H) = 2 \). Then by Theorem 2.8, \( G + H = K_2 + H_1 \), where \( H_1 \) is a non-complete graph. Let \( V(K_2) = \{x, y\} = S \), then \( S \) is a SCDS in \( G + H \). Consider the following cases:

Case 1: Suppose \( x, y \in V(G) \)

Then \( G = K_2 + J \), where \( J \) is a non-complete subgraph of \( G \). Hence by Theorem 2.8, \( \gamma_{sc}(G) = 2 \).

Case 2: Suppose \( x, y \in V(H) \)

As in case 1, \( \gamma_{sc}(H) = 2 \).
Case 3: Suppose \( x \in V(G) \) and \( y \in V(H) \)

Let \( z \in V(G) \setminus \{x\} \). Since \( G + H = K_2 + H_1 \), \( xz \in E(G) \). Hence \( \{x\} \) is a dominating set in \( G \). It follows that \( \gamma(G) = 1 \). Similarly, \( \gamma(H) = 1 \).

For the converse, suppose that \( \gamma_{sc}(G) = 2 \). Then by Theorem 2.8, \( G = K_2 + J \), where \( J \) is a non-complete graph. Thus, \( G + H = (K_2 + J) + H = K_2 + H_1 \), where \( H_1 = J + H \) is non-complete. It follows from Theorem 2.8 that \( \gamma_{sc}(G + H) = 2 \). Similarly, \( \gamma_{sc}(G + H) = 2 \) if \( \gamma_{sc}(H) = 2 \) holds. Suppose (iii) holds. Let \( \{x\} \) and \( \{y\} \) be a dominating set of \( G \) and \( H \), respectively. Then \( G = \langle x \rangle + J \) and \( H = \langle y \rangle + K \) where \( J \) and \( K \) are non-complete graphs.

Then, \( G + H = (\langle x \rangle + J) + (\langle y \rangle + K) = K_2 + H_1 \), where \( V(K_2) = \{x, y\} \) and \( H_1 = J + K \) is a non-complete graph. Hence by Theorem 2.8, \( \gamma_{sc}(G + H) = 2 \).

\( \square \)

**Theorem 2.17** Let \( G \) and \( H \) be non-complete connected graph such that \( \gamma_{sc}(G + H) \neq 2 \). Then \( \gamma_{sc}(G + H) = 3 \) if and only if one of the following holds:

(i) \( \gamma_{sc}(G) = 3 \) or \( \gamma_{sc}(H) = 3 \)

(ii) \( \gamma(G) = 2 \) or \( \gamma(H) = 2 \).

**Proof**: Suppose that \( \gamma_{sc}(G + H) = 3 \). Let \( S = \{x, y, z\} \) be a SCDS in \( G + H \). Consider the following cases:

**Case 1**: Suppose \( S \subseteq V(G) \)

Then \( S \) is an SCDS in \( G \). It follows that \( \gamma_{sc}(G) = 3 \)

**Case 2**: If \( S \subseteq H \)

Then \( S \) is an SCDS in \( H \) and \( \gamma_{sc}(H) = 3 \).

**Case 3**: Suppose \( V(G) \cap S \neq \emptyset \) and \( V(H) \cap S \neq \emptyset \)

Let \( w \in V(G) \setminus \{x, y\} \). Suppose that \( w \notin N_G(X) \). Then \( wz \in E(G + H) \).

Hence, \( \{S \setminus \{z\}\} \cup \{w\} \) is not a CDS, which is contrary to the assumption about \( S \). Therefore, \( N_G[X] = V(G) \), that is, \( X \) is a DS in \( G \).

Since \( G \) is non-complete, \( \gamma(G) = 2 \).

For the converse, suppose (ii) holds, say \( \gamma(G) = 2 \). Let \( X = \{a, b\} \) be a DS in \( G \). Pick \( c \in V(H) \) and set \( S = \{a, b, c\} \). Let \( z \in V(G + H) \setminus S \). Then \( z \in V(G) \) or \( z \in V(H) \). If \( z \in V(G) \), then \( z \in N_G(X) \), say \( az \in E(G + H) \), and \( S = \{b, z, c\} \) is a CDS in \( G + H \). If \( z \in V(H) \), then \( az \in E(G + H) \) and \( S = \{b, z, c\} \) is a CDS in \( G + H \).

Thus, \( S \) is an SCDS in \( G + H \). Therefore \( \gamma_{sc}(G + H) = 3 \). We obtain the same conclusion if we assume that \( \gamma(H) = 2 \).

Suppose (i) holds, say \( \gamma_{sc}(H) = 3 \). Let \( S = \{a, b, c\} \) be a SCDS in \( H \). Then, clearly \( S \) is a SCDS in \( G + H \). Therefore, \( \gamma_{sc}(G + H) = 3 \).

\( \square \)

**Theorem 2.18** Let \( G \) be a non-trivial connected graph and let \( H \) be any graph. Then \( C \subseteq V(G \circ H) \) is a secure connected dominating set in \( G \circ H \) if and only if \( C = V(G) \cup (\bigcup_{v \in V(G)} S_v) \), where each \( S_v \) is a dominating set in \( H_v \).
Secure connected domination in a graph

Proof: Suppose that \( C \subseteq V(G \circ H) \) is a secure connected dominating set in \( G \circ H \) and let \( v \in V(G) \). Choose \( w \in V(G) \setminus \{v\} \). Since \( C \) is a DS, \( V(v + H^v) \cap C \neq \emptyset \) and \( V(w + H^w) \cap C \neq \emptyset \). Since \( (C) \) is connected, \( v \in C \). Let \( S_v = V(H^v) \cap C \) and \( x \in V(H^v) \setminus S_v \). Since \( C \) is an SCDS of \( G \circ H \), there exists \( y \in C \) such that \( xy \in E(G \circ H) \) and \( (C \setminus \{y\}) \cup \{x\} \) is a CDS. Note that since \( (\langle C \setminus \{v\} \rangle \cup \{x\}) \) is not connected in \( G \circ H \), it follows that \( y \neq v \). Thus, \( y \in S_v \), showing that \( S_v \) is a DS in \( H^v \).

Conversely, suppose that \( C = V(G) \cup (\cup_{v \in V(G)} S_v) \). Then, clearly, \( (C) \) is a CDS in \( G \circ H \). Let \( x \in V(G \circ H) \setminus C \) and let \( v \in V(G) \) such that \( x \in V(H^v) \setminus S_v \). Since \( S_v \) is a DS in \( H^v \), there exists \( z \in S_v \) such that \( xz \in E(G \circ H) \). Also, \( (C \setminus \{x\}) \cup \{z\} \) is a CDS in \( G \circ H \). Therefore, \( C \) is an SCDS in \( G \circ H \). \( \square \)

Corollary 2.19 Let \( G \) be a non-trivial connected graph of order \( m \) and let \( H \) be any graph of order \( m \). Then \( \gamma_{sc}(G \circ H) = m(1 + \gamma(H)) \).

Proof: Let \( S \) be a \( \gamma \)-set of \( H \). For each \( v \in V(H) \), choose \( S_v \subseteq V(H^v) \) such that \( S_v \cong S \). Then, by Theorem 2.18, \( C = V(G) \cup (\cup_{v \in V(G)} S_v) \) is an SCDS in \( G \circ H \). Hence, \( \gamma_{sc}(G \circ H) \leq m + \sum |S_v| = m + m\gamma(H) = m(1 + \gamma(H)) \). On the other hand, if \( C \) is a \( \gamma_{sc} \)-set in \( G \circ H \), then \( C = V(G) \cup (\cup S_v^*) \), where each \( S_v^* \) is a DS in \( H^v \), by Theorem 2.18. Thus, \( \gamma_{sc}(G \circ H) = |C| = m + \sum |S_v^*| \geq m + m\gamma(H) = m(1 + \gamma(H)) \). This proves the desired equality. \( \square \)

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References


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