The Vector Valued Sumudu Transform

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Abstract

In this paper, the Sumudu transform of vector valued function is introduced and a number of results have been derived. This is a generalized form of Sumudu transform which itself has nice properties.

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1 Introduction

The Sumudu transform is derived from the classical Fourier transform by Watugala [5] who used it to solve differential equations and integral equations in the time domain.

1.1 Definition:

Taking a clue from the theory of $\mathbb{R}^n$ we define vector-valued Sumudu transform of a vector-valued function $f(t_1,t_2,\ldots,t_n) = f(t)$. Let $C(\mathbb{R}^n)$ be the space
of all continuous vector valued functions on $\mathbb{R}^n$ then vector valued Sumudu transform of $f(t) \in C(\mathbb{R}^n)$ is denoted and defined as

$$S_n[f(t); u] = (u_1, u_2, ..., u_n)^{-1} \int_{t} e^{t^{-1}u^{-1}} f(t) P_n(dt) = G(u). \quad (1.1.1)$$

Where $P_n(dt) = \prod_{i=1}^{n} dt_i$, $I = \mathbb{R}^n$, $u^{-1} = (u_1^{-1}, u_2^{-1}, ..., u_n^{-1})$, $t = (t_1, t_2, ..., t_n)$. When $n=1, 2, 2$, the definition reduces as given by [1, 5].

2 Main Results

In this section we apply vector valued Sumudu transform on some elementary vector valued functions and later in the section we use symmetric polynomials to derive some very interesting properties of vector valued Sumudu transform followed by a corollary and an example.

2.1 Vector valued Sumudu transform of elementary functions:

In this section, we will derive vector-valued Sumudu transform of some basic functions.

2.1 Theorem

Let $f(t) \in C(\mathbb{R}^n)$. We denote $\sqrt{t} = (t_1^{1/2}, t_2^{1/2}, ..., t_n^{1/2})$ so that

$$e^{\sqrt{t} u} = e^{-(t_1 u_1 + t_2 u_2 + \cdots + t_n u_n)}$$

Then

$$S_n[e^{\sqrt{t} u}; u] = \left( \prod_{i=1}^{n} \frac{1}{1+u_i} \right) \quad (2.1.1)$$

Proof:

Using the definition (1.1.1) and arranging, we have

$$S_n[e^{\sqrt{t} u}; u] = (u_1, u_2, ..., u_n)^{-1} \int_{t} e^{\left( \frac{u_1}{u_1} + \frac{u_2}{u_2} + \cdots + \frac{u_n}{u_n} \right) \sqrt{t}} e^{-(t_1 u_1 + t_2 u_2 + \cdots + t_n u_n)} \left( \frac{1}{u_i} \int_{0}^{t} e^{-\left( \frac{1}{u_i} t \right)} dt_i \right) dt_2...dt_n \quad (2.1.2)$$

Integrating the inner integral we get
The vector valued Sumudu transform

\[
\frac{1}{u_1} \int_0^\infty e^{-\left(\frac{1}{u_1}\right)t_1} \, dt_1 = \frac{1}{u_1} \left(\frac{1}{\left(\frac{1}{u_1}+1\right)}\right) = \frac{1}{1+u_1}
\]

(2.1.3)

Plugging (2.1.3) into (2.1.2) and arranging we have

\[
S_n \left[ e^{-[k^T \cdot u]} \right] = \left(\frac{1}{1+u_1}\right) \left(\frac{1}{1+u_2}\right) \int_0^\infty \cdots \int_0^\infty e^{\left(\frac{t_{u_1+1}}{u_1}\right) \cdots \frac{t_{u_n+1}}{u_n}} e^{-\left(\frac{1}{u_1}\right)t_1} e^{-\left(\frac{1}{u_2}\right)t_2} \, dt_1 \cdots dt_n
\]

Again integrating the inner integral as we did before, we get from the last equation

\[
S_n \left[ e^{-[k^T \cdot u]} \right] = \left(\frac{1}{1+u_1}\right) \left(\frac{1}{1+u_2}\right) \int_0^\infty \cdots \int_0^\infty e^{\left(\frac{t_{u_1+1}}{u_1}\right) \cdots \frac{t_{u_n+1}}{u_n}} e^{-\left(\frac{1}{u_1}\right)t_1} e^{-\left(\frac{1}{u_2}\right)t_2} \, dt_1 \cdots dt_n.
\]

Continuing in this way, we get the desired result (2.1.1).

2.2 Theorem

Let \( f(t) \in C\left(\mathbb{R}^n\right) \). We define \( \sqrt{t} = \frac{t^{1/2}}{2} = \left(t_1^{1/2}, t_2^{1/2}, \ldots, t_n^{1/2}\right) \) such that \( f(t) = t_1 \cdots t_n e^{-[k^T \cdot u]} \).

Then

\[
S_n \left[ t_1 \cdots t_n e^{-[k^T \cdot u]} \right] = \prod_{i=1}^n \left(\frac{u_i}{(1-u_i)^2}\right)
\]

(2.2.1)

Proof:

By definition (1.1.1) we can write

\[
S_n \left[ t_1 \cdots t_n e^{-[k^T \cdot u]} \right] = (u_2 \cdots u_n)^{-1} \int_0^\infty \cdots \int_0^\infty t_1 \cdots t_n e^{-\left(\frac{1}{u_1}\right)^2 \cdots \frac{1}{u_n}}
\]

\[
\times \left(\frac{1}{u_1} \int_0^\infty t_1 e^{-\left(\frac{1}{u_1}\right)^2} \, dt_1\right) dt_2 \cdots dt_n.
\]

(2.2.2)

Integrating the inner integral by parts with respect to \( t_1 \) we get

\[
\frac{1}{u_1} \int_0^\infty t_1 e^{-\left(\frac{1}{u_1}\right)^2} \, dt_1 = \frac{u_1}{(1-u_1)^2}.
\]

(2.2.3)

Substituting (2.2.3) into (2.2.2) we have

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\[ S_n \left[ t_1 \ldots t_n e^{\int_{u_1}^{u} u dt} \right] = \frac{u_1}{(1-u_1)^3} (u_2 \ldots u_n) \int_0^\infty \ldots \int_0^\infty t_2 \ldots t_n e^{-\frac{(u_1, \ldots, u_n)}{u_1 \ldots u_n}} dt_2 \ldots dt_n \]  \hspace{1cm} (2.2.4)

Integrating the remaining integral by parts in a similar manner as we did above we can write

\[ \frac{1}{u_i} \int_0^\infty t e^{-\frac{(u_1, \ldots, u_n)}{u_i}} dt = \frac{u_i}{(1-u_i)^3}, \quad 2 \leq i \leq n. \] \hspace{1cm} (2.2.5)

Now from (2.2.3) together with (2.2.5) we have the required result (2.2.1).

The next theorem ensures that we can apply Sumudu transform on integral of a function.

2.3 Theorem

Let \( f(t) \in C(\mathbb{R}^n) \) and \( W_n(t_1, t_2, \ldots, t_n) \) denote the \( n \) times definite integral of \( f \)

\[ W_n(t_1, t_2, \ldots, t_n) = \int_0^{t_1} \ldots \int_0^{t_2} f(\tau_1, \tau_2, \ldots, \tau_n) d\tau_1 d\tau_2 \ldots d\tau_n. \]

Then

\[ S_n \left[ W_n(t_1, t_2, \ldots, t_n) \right] = u_1 u_2 \ldots u_n G_n(u_1, u_2, \ldots, u_n). \] \hspace{1cm} (2.3.1)

Proof:

Using the definition of vector valued Sumudu transform we can write

\[ S_n \left[ W_n(t_1, \ldots, t_n) \right] = (u_1 \ldots u_n)^{-\frac{1}{u_1}} \int_0^\infty \ldots \int_0^\infty e^{-\frac{(u_1, \ldots, u_n)}{u_i}} \int_0^{t_1} \ldots \int_0^{t_n} f(\tau_1, \ldots, \tau_n) d\tau_1 \ldots d\tau_n \] \hspace{1cm} (2.3.2)

Rearranging the right hand side (2.3.2) and the inner most integral and integrating by parts with respect to \( t_i \), we get

\[ \int_0^\infty e^{-\frac{u}{u_i}} \left( \int_0^{t_i} f(\tau_1, \ldots, \tau_n) d\tau_1 \right) dt_i = u_i G(u_1, \tau_2, \ldots, \tau_n) \] \hspace{1cm} (2.3.3)

We used the notation \( G(u_1, \tau_2, \ldots, \tau_n) \) for the Sumudu transform of \( f(t_1, \tau_2, \ldots, \tau_n) \) with respect to \( t_1 \) while treating \( \tau_i, \quad 2 \leq i \leq n \) as constants. Now
substituting (2.3.3) into (2.3.2)

\[ S_n \left[ W_n \left( t_1, \ldots, t_n \right) \right] = u_1 \left( u_3 \ldots u_n \right)^{-1} \int_0^\infty \cdots \int_0^\infty e^{\frac{t_1}{u_1} + \cdots + \frac{t_n}{u_n}} \left[ \int_0^\infty \int_0^{t_1} \frac{1}{u_2} e^{\frac{t_2}{u_2}} \cdots \int_0^{t_2} \frac{1}{u_3} e^{\frac{t_3}{u_3}} \cdots \int_0^{t_n} \frac{1}{u_n} e^{\frac{t_n}{u_n}} \right] dt_1 \cdots dt_n \]

(2.3.4)

Again consider the inner integral and integrating with respect \( t_2 \) we have

\[ \frac{1}{u_2} \int_0^{t_2} e^{\frac{t_2}{u_2}} \left( \int_0^{t_2} G(u_1, \tau_2, \ldots, \tau_n) d\tau_2 \right) d\tau_2 = u_2 G(u_1, u_2, \ldots, \tau_n) \]

(2.3.5)

Substituting (2.3.5) into (2.3.4) we gain

\[ S_n \left[ W_n \left( t_1, \ldots, t_n \right) \right] = u_1 u_2 \left( u_3 \ldots u_n \right)^{-1} \int_0^\infty \cdots \int_0^\infty e^{\frac{t_1}{u_1} + \cdots + \frac{t_n}{u_n}} \left[ \int_0^{t_1} \cdots \int_0^{t_n} G(u_1, u_2, \ldots, \tau_n) \right] d\tau_1 \cdots d\tau_n dt_1 \cdots dt_n. \]

(2.3.6)

Continuing in same argument, we eventually get (2.3.4).

Now, we apply mathematical induction to prove the above result. For \( n = 1 \), (2.3.4) holds. Now suppose that (2.3.1) holds for any integer \( n, n > 1 \). We show that it will also hold for \( n+1 \). Consider

\[ S_{n+1} \left[ W_n \left( t_1, t_2, \ldots, t_{n+1} \right) \right] = S_{n+1} \left[ \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_n} f(\tau_1, \tau_2, \ldots, \tau_n) d\tau_1 d\tau_2 \cdots d\tau_n \right] \]

\[ = S_{n+1} \left[ \int_0^{t_1} \cdots \int_0^{t_n} W_n \left( t_1, t_2, \ldots, t_n \right) d\tau_n \right] \]

\[ = u_{n+1} S_n \left[ W_n \left( t_1, t_2, \ldots, t_n \right) \right] (u_{n+1}) \]

\[ = u_{n+1} u_1 \cdots u_n F(u_1, \ldots, u_n)(u_{n+1}). \]

Therefore, we get

\[ S_{n+1} \left[ W_n \left( t_1, t_2, \ldots, t_{n+1} \right) \right] = u_1 \cdots u_n u_{n+1} G(u_1, u_2, \ldots, u_n, u_{n+1}). \]

So the result (2.3.1) is true for all integral values of \( n \).

2.4 Vector valued Sumudu Transform involving Symmetric Polynomials
2.5 Definition:

Let \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) and \( \mathbf{s} = (s_1, s_2, \ldots, s_n) \) for real \( n \) and let \( P_k(\mathbf{t}) \) be the \( k \)-th symmetric polynomial in the component \( t^k \) of \( \mathbf{t} \). Then

\[
P_k(t^n) = t_1^n + t_2^n + \ldots + t_n^n,
\]

\[
P_n(t^n) = t_1 t_2^n \ldots t_n^n.
\]

With the above notation we see that (1.1.1) can be written as

\[
S_n\left[ f(t); (u) \right] = G(u) = P_n^{-1}(u) \int_0^\infty \cdots \int_0^\infty e^{-n(x^2)} f(t) dt_1 \ldots dt_n
\]

(2.5.1)

2.6 Theorem

Let \( f(t) \in C(\mathbb{R}^n) \) and \( F(u) \) be the Sumudu transform of \( f(t) \) and

\[
G(u) = S\left[ t^{-\frac{n}{2}} F(t) \right],
\]

(2.6.1)

\[
H(u) = S\left[ t f(t^4) \right].
\]

(2.6.2)

Then

\[
S_n\left[ P_n\left( t^{-\frac{n}{2}} \right) P_n^{-2}(t^{-1}) G\left( P_n^{-2}(t^{-1}) \right) \right] = \frac{8\sqrt{2\pi}^{(n+1)/2} P_n(u^{1/2})}{P_n\left( u^{1/2} \right)} H\left( 2\sqrt{2} P_n^{-1}\left( u^{1/2} \right) \right)
\]

(2.6.3)

Proof:

We have, by definition of Sumudu transform and by (1.1.1) for \( n = 1 \)

\[
u G(u) = 2 \int_0^\infty f(x) t^{-\frac{3}{2}} e^{-\frac{x}{t^{3/2}}} dt dx.
\]

(2.6.3)

The inner integral on the right can be easily solved and is equal to

\[
\int_0^\infty t^{-\frac{3}{2}} e^{-\frac{x}{t^{3/2}}} dt = \frac{\sqrt{\pi} e^{-\frac{\sqrt{x}}{\sqrt{t}}}}{\sqrt{x}}
\]

So (2.6.3) reduces to

\[
u G(u) = \sqrt{\pi} \int_0^\infty f(x) e^{-\frac{\sqrt{x} \sqrt{u}}{\sqrt{t}}/\sqrt{x}} dx.
\]

(2.6.4)

Now setting \( x = v^4 \Rightarrow dx = 4v^3 dv \) we obtain
The vector valued Sumudu transform

\[ uG(u) = 4\sqrt{\pi} \int_0^\infty vf(v^4)e^{-\frac{\beta^2}{v^4}}dv. \]  

(2.6.5)

Now, replacing \( u \) by \( P_1^{-2}(t^{-1}) \) and multiplying both sides by \( e^{i\beta}\int P_n(t^{-\frac{3}{2}}) \), and integrating \( n \) times w.r.t. \( t_1, t_2, \ldots, t_n \) From 0 to \( \infty \), the LHS of (2.6.5) becomes

\[ J_n(u) = P_n(u)S_n\left[P_n\left(t^{-\frac{3}{2}}\right)P_1^{-2}\left(t^{-1}\right)G\left(P_1^{-2}\left(t^{-1}\right)\right)\right]. \]  

(2.6.6)

So (2.6.5) takes the form

\[ J_n(u) = uG(u) = 4\sqrt{\pi} \int_0^\infty \cdots \int_0^\infty e^{i\beta}\int P_n(t^{-\frac{3}{2}})e^{-i\beta^2\eta(v^4)}dvdt_1\ldots dt_n \]  

(2.6.7)

Changing the order of integration on right hand side, we can write

\[ J_n(u) = 4\sqrt{\pi} \int_0^\infty \cdots \int_0^\infty t_1^{-\frac{3}{2}}\cdots t_n^{-\frac{3}{2}} e^{-\frac{\beta^2}{4} \left(\frac{1}{t_1} + \ldots + \frac{1}{t_n}\right)} dt_1\ldots dt_n \]  

\[ \int_0^\infty \cdots \int_0^\infty t_1^{-\frac{3}{2}}\cdots t_n^{-\frac{3}{2}} e^{-\frac{\beta^2}{4} \left(\frac{1}{t_1} + \ldots + \frac{1}{t_n}\right)} \left\{ \int_0^\infty \cdots \int_0^\infty t_1^{-\frac{3}{2}}\cdots t_n^{-\frac{3}{2}} e^{-\frac{\beta^2}{4} \left(\frac{1}{t_1} + \ldots + \frac{1}{t_n}\right)} dt_1\ldots dt_n \right\} dt_1\ldots dt_n \]  

Evaluating the \( n \) integral on the right hand side of the last equation by using the result

\[ \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{\beta^2}{4} \left(\frac{1}{t}\right)} dt = \frac{\pi}{\sqrt{\beta}} e^{-2\sqrt{\beta}t} \]  

we obtain

\[ J_n(u) = 4\pi \int_0^\infty P_n\left(u\frac{1}{2}\right)^\frac{n+1}{2} e^{-\frac{\beta^2}{4} P_1^{-1}\left(u\frac{1}{2}\right)} dv \]  

(2.6.8)

Now using (2.6.2) we can (2.6.8) as below

\[ J_n(u) = 8\sqrt{2\pi} \int_0^\infty P_n\left(u\frac{1}{2}\right)^\frac{n+1}{2} e^{-\frac{\sqrt{2}}{4} P_1^{-1}\left(u\frac{1}{2}\right)} dv \]  

(2.6.9)

Finally, using (2.6.9) along with (2.6.6), we get the desired result (2.6.3).

2.6.1 Corollary

Letting \( n = 2, t = (x, y), u = (u, v) \) in (2.6.3), we get

\[ S_{2} \left[\frac{(xy)^{\frac{1}{2}}}{(x+y)^{2}} G\left(\frac{(xy)^{\frac{1}{2}}}{(x+y)^{2}}; u, v\right) \right] = 8\sqrt{2\pi} \frac{\sqrt{uv}}{\left(\sqrt{u} + \sqrt{v}\right)^{2}} H\left(\frac{2\sqrt{2}}{\sqrt{u} + \sqrt{v}}\right). \]
2.6.2 Example

If \( f(t) = \ln(at) \), \( a > 0 \), then we observe that

\[
F(u) = \ln(au) + \int_0^\infty e^{-t} \ln(t) \, dt,
\]

\[
G(u) = \frac{1}{\sqrt{\pi}} \left[ \sqrt{\pi} (\ln(au) - \gamma) + \Gamma\left(\frac{1}{2}\right) \right],
\]

where we have used well known gamma function \( \Gamma(n) = \int_0^\infty e^{-t} t^{n-1} \, dt \) and \( \gamma \) is the Euler’s constant.

\[
H(u) = u \left[ \ln\left(au^2\right) - 4\gamma \right].
\]

Now, using the corollary (2.6.1), we can write as

\[
S_2 \left[ \frac{(x+y)^{\nu/2}}{(x+y)} \sqrt{\pi} \left( \ln \left( \sin \frac{xy}{x+y} \right) \right)^2 + \Gamma\left(\frac{1}{2}\right) \right]_{u,v} = 32\pi^{\nu/2} \frac{\sqrt{uv}}{\left(\sqrt{u} + \sqrt{v}\right)^2} \left[ \ln \left( \frac{64a}{\left(\sqrt{u} + \sqrt{v}\right)^2} \right) - 4\gamma \right].
\]

References


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