Some Extremal Properties of a Generalised Close-to-Convex Function

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Abstract

Let $G_{St}(\alpha, \delta)$ denotes the class of function $f$, $f(0) = f''(0) - 1 = 0$ and satisfying
\[
\text{Re}\left\{ e^{in} \frac{zf'(z)}{f(z) - f(-z)} \right\} > \delta,
\]
in $E = \{z : |z| < 1\}$ for $|\alpha| < \pi$, $0 \leq \delta < 1$ and $\cos \alpha > \delta$. In this paper, we determine the bounds for $\arg f'(z)$ of $G_{St}(\alpha, \delta)$.

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1 Introduction

Let \( A \) denote the class of function given by
\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots = z + \sum_{n=2}^{\infty} a_n z^n
\] (1.1)
that are analytic in \( E = \{ z : |z| < 1 \} \). We define \( G_{S\alpha}(\alpha, \delta) \) as the class of normalise function \( f \in A \) satisfying the condition
\[
\Re \left\{ e^{i\alpha} \frac{zf''(z)}{f(z) - f(-z)} \right\} > \delta, \quad (z \in E)
\]
where \( |\alpha| < \pi \), \( 0 \leq \delta < 1 \) and \( \cos \alpha > \delta \). These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [4] in 1959. We also define subclass of \( A \) consisting of function that are Univalent, Starlike, Convex and Close-to-Convex denotes by \( S, St, K, \) and \( C \) respectively. Based on Goodman [1], the class of \( St, K, \) and \( C \) are defined as follows

**Definition 1.1** Let \( f \) be given by (1.1). Then \( f \in St \) if and only if
\[
\Re \left\{ \frac{zf''(z)}{f(z)} \right\} > 0, \quad z \in E.
\]

**Definition 1.2** Let \( f \) be given by (1.1). Then \( f \in K \) if and only if
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in E.
\]

**Definition 1.3** Let \( f \) be given by (1.1). Then \( f \in C \) if and only if for \( z \in E \) and \( -\frac{\pi}{2} < \beta < \frac{\pi}{2} \) there exist \( g(z) \in K \) such that
\[
\Re \left\{ e^{i\beta} \frac{f''(z)}{g'(z)} \right\} > 0, \quad z \in E.
\]

Recently, Yahya et al., [2][3] defined \( G_{S\alpha}(\alpha, \delta) \) as the class of functions \( f \in S \) satisfy
\[
\Re \left\{ e^{i\alpha} \frac{zf''(z)}{g(z)} \right\} > \delta, \quad \left( z \in E; |\alpha| \leq \pi; \cos \alpha > \delta; g(z) = \frac{z}{1-z^2} \right)
\]
and shows some extremal properties such as representation theorem, extreme points, bound of \( a_n \), upper and lower bounds for \( \Re f' \), \( \Im f' \) and distortion.
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theorem for this class of functions. Our purpose on this paper is to obtain others
basic properties such as bounds for \( \arg f'(z) \) of \( G_{S_\alpha}(\alpha, \delta) \).

2 Main Result

In Yahya et al., [3] the Centre and Radius and Distortion Theorem for
\( G_{S_\alpha}(\alpha, \delta) \) are given by following lemma.

**Lemma 2.1** Let \( f \in G_{S_\alpha}(\alpha, \delta) \), then \( f' \) maps \(|z| \leq r\) into the disc \( \Delta_r \), with
centre
\[
-e^{-ia}(e^{-ia} - 2\delta) \frac{1}{1-r^4} + \frac{2A_{a\delta}e^{-ia}}{(1-r^2)(1-r^4)}
\]
and radius
\[
\frac{2A_{a\delta}r}{(1-r^2)(1-r^4)}.
\]

**Lemma 2.2** Let \( f \in G_{S_\alpha}(\alpha, \delta) \), then
\[
|f'(z)| \leq C(r) + \frac{2A_{a\delta}r}{(1-r^2)(1-r^4)}
\]
where
\[
C(r) = \left[1 + \frac{4r^2A_{a\delta}}{1-r^2}\left(A_{a\delta} + \delta\right)\right]^{\frac{1}{2}} \left[\frac{1}{1-r^4}\right]
\]
(2.1)
and the bound is sharp for any extreme points of \( G_{S_\alpha}(\alpha, \delta) \).

**Theorem 2.3** Let \( f \in G_{S_\alpha}(\alpha, \delta) \) and put \( x(r) = \frac{2r^2A_{a\delta}}{(1-r^2)(1-r^4)} \), \((0 \leq r < 1)\). Let
\[
r_0 = \begin{cases} 
1, & \delta \geq 0 \\
\frac{1}{\sqrt{1-4\delta A_{a\delta}}}, & \delta < 0
\end{cases}
\]
Then, for \( 0 < |z| = r < r_0 \) and for suitable determination of argument
\[
|\arg f'(z) + \alpha - \phi_a(x(r))| \leq \sin^{-1} \frac{2rA_{a\delta}}{(1-r^2)(1-r^4)C(r)}
\]
where \( \phi_a(x) \) is defined on \( [0,x(r_o)] \) as above, and \( C(r) \) is given by (2.1).

**Proof** Following work by Soh and Mohamad [5] and from Lemma 2.1, the distance of the centre from the origin is greater than the radius. Thus,

\[
\left| \frac{2A_{ao\delta}}{1-r^2(1-r^4)} - \frac{e^{-ia} - 2\delta}{1-r^4} \right| > \frac{2rA_{ao\delta}}{1-r^2(1-r^4)}.
\]

From the above inequality, we have

\[
\left( \frac{2A_{ao\delta}}{1-r^2(1-r^4)} + \frac{2\delta}{1-r^4} \right)^2 - 2\cos \alpha \left( \frac{2A_{ao\delta}}{1-r^2(1-r^4)} + \frac{2\delta}{1-r^4} \right) + \frac{\cos^2 \alpha}{1-r^4} + \frac{\sin^2 \alpha}{1-r^4} - \frac{4r^2 A_{ao\delta}^2}{(1-r^2(1-r^4))} > 0,
\]

and hence

\[
\frac{4\delta A_{ao\delta}}{1-r^2(1-r^4)} - \frac{4\delta A_{ao\delta}}{1-r^2(1-r^4)} + \frac{1}{1-r^4} > 0,
\]

This inequality holds for all \( r \) in \( [0,1) \) if \( \delta \geq 0 \) and for \( 0 \leq r < \frac{1}{\sqrt{1-4\delta A_{ao\delta}}} \) if \( \delta < 0 \). This establishes the restriction on \( |z| \) in the statement of the theorem.

From Lemma 2.2, we have

\[
\left| f''(z) - \left( - \frac{-e^{-ia}e^{-ia} - 2\delta}{1-r^4} + \frac{2A_{ao\delta}e^{-ia}}{1-r^2(1-r^4)} \right) \right| \leq \frac{2A_{ao\delta}}{1-r^2(1-r^4)}.
\]

and with \( H(r) \) given as \( H(r) = -\frac{-e^{-ia}e^{-ia} - 2\delta}{1-r^4} + \frac{2A_{ao\delta}e^{-ia}}{1-r^2(1-r^4)} \) and \( C(r) = |H(r)| \), we deduce

\[
|\arg f''(z) - \arg H(r)| \leq \sin^{-1} \left( \frac{2rA_{ao\delta}}{1-r^2(1-r^4)C(r)} \right)
\]

(2.3)

also,

\[
\arg H(r) = \arg \left( \frac{2A_{ao\delta}}{1-r^2(1-r^4)} - \frac{e^{-ia} - 2\delta}{1-r^4} \right)
\]

\[
= -\alpha + \arg \left( \frac{e^{ia}}{1-r^4} + \frac{2r^2 A_{ao\delta}}{1-r^2(1-r^4)} \right)
\]
Put \( x(r) = \frac{2r^2A_{\alpha \delta}}{(1-r^2)(1-r^4)} \), then \( \arg H(r) = -\alpha + \phi_\alpha(x(r)) \) and the desired result follows using (2.3).

We prove the following theorem that is similar to the previous result, but feature \( \arg(f'(z) + k) \) for some real number \( k \) instead of \( \arg f'(z) \) with a restricted range of \( |z| \).

**Theorem 2.4**  
Let \( f \in G_{\alpha, \delta} \), where \( |\alpha| \neq \frac{\pi}{2} \). Put \( x(r) = \frac{2r^2A_{\alpha \delta}}{(1-r^2)(1-r^4)} \), 

\[
(0 \leq r < 1) \quad \text{and let} \ k \ \text{be a real number such that} \\
k \cos \alpha + \delta \left( \frac{1}{1-r^4} \right) > 0.
\]

Then

\[
|\arg(f'(z) + k) + \alpha - \phi_\alpha(x(r))| \leq \sin^{-1} \frac{2rA_{\alpha \delta}}{(1-r^2)(1-r^4)C_1(r)}
\]

where \( \phi_\alpha \) is defined on \( [0, \infty) \) as the continuous argument of \( \left( k + \frac{1}{1-r^4} \right)e^{i\alpha} + x \) and

\[
C_1(r) = \sqrt{\frac{4A_{\alpha \delta}r^2}{(1-r^2)(1-r^4)} + \frac{A_{\alpha \delta}}{(1-r^2)(1-r^4)} + k \cos \alpha + \frac{\delta}{1-r^4} + \left( \frac{1}{1-r^4} + k \right)^2}.
\]

**Proof**  
Let \( |\alpha| \neq \frac{\pi}{2} \), and let \( k \) satisfy \( k \cos \alpha + \delta \left( \frac{1}{1-r^4} \right) > 0 \). From (2.2), we have,

\[
|f'(z) + H(r) + k| \leq \frac{2rA_{\alpha \delta}}{(1-r^2)(1-r^4)}
\]

where

\[
H(r) = -e^{-i\alpha} \left( e^{-i\alpha} - 2\delta \right) + 2A_{\alpha \delta} e^{-i\alpha} = \frac{1}{1-r^4} + \frac{2A_{\alpha \delta}r^2 e^{-i\alpha}}{(1-r^2)(1-r^4)}.
\]

Hence,
\[ \arg f'(z + k) - \arg (H(r) + k) \leq \sin^{-1} \frac{2r A_{\alpha \delta}}{(1 - r^2)(1 - r^4)} C_1(r) \] (2.4)

where

\[ C_1(r) = |H(r) + k| = \left| \frac{1 - r^2 + 2 A_{\alpha \delta} r^2 \cos \alpha + k(1 - r^2) (1 - r^4)}{(1 - r^2)(1 - r^4)} + \frac{2 A_{\alpha \delta} r^2 \sin \alpha}{(1 - r^2)(1 - r^4)} \right| \]

\[ = \frac{4 A_{\alpha \delta} r^2}{(1 - r^2)(1 - r^4)} \left[ \frac{A_{\alpha \delta}}{(1 - r^2)(1 - r^4)} + k \cos \alpha + \frac{\delta}{1 - r^4} \right] + \left( \frac{1}{1 - r^4} + k \right)^2. \]

Now,

\[ \arg (H(r) + k) = \arg \left[ e^{-i\alpha} \left( \frac{2 A_{\alpha \delta}}{(1 - r^2)(1 - r^4)} - \frac{e^{-i\alpha}}{(1 - r^4)} \right) + k \right] \]

\[ = -\alpha + \varphi_{\alpha}(x(r)) \]

and with (2.4) this completes the proof.

References


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