International Journal of Mathematical Analysis
Vol. 8, 2014, no. 38, 1893 - 1906
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2014.47218

Some Generalized Inequalities for Differentiable Harmonically Quasi-Convex Functions

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Abstract

In this paper, by setting up a generalized integral identity for differentiable functions, the author obtain some integral inequalities for functions whose derivatives in the absolute value are differentiable harmonically quasi-convex.

Mathematics Subject Classification: 26A51, 26D15

Keywords: Hermite-Hadamard type inequality, Hölder's inequality, Harmonically quasi-convexity, mean power integral inequality.

1 Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \to R$ be a convex function and $a, b \in I$ with $a < b$. Then following double inequalities hold:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Hermite-Hadamard's inequalities for convex functions, (α, m)-convex functions, and s-convex functions in the second sense have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [1] and [5]-[12] and references therein.

The notion of quasi-convex functions generalize the notion of convex functions. More precisely, a function $f : [a, b] \to R$ is said to be quasi-convex on $[a, b]$ if the inequality

$$f\left(tx + (1-t)y\right) \leq \sup\{f\left(x\right), f\left(y\right)\},$$

holds, for all $x, y \in [a, b]$ and $t \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex [4].

Let us recall some definitions of several kinds of convex functions:

Definition 1. Let I be an interval in R. Then $f : I \to R$ is said to be convex on I if the inequality

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right) \tag{2}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. Let I be an interval in $R_+ \setminus \{0\}$. A function $f : I \to R$ is said to be harmonically convex on I if the inequality

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq tf\left(y\right) + (1-t)\,f\left(x\right) \tag{3}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

For some results which generalize, improve and extend Hermite-Hada -mard type inequalities concerning harmonically convex functions, please see [2, 3] and references thereine. In [2, 3], Imdat Işcan established the following Hermite-Hadamard-like type inequality for harmonically convex functions:

Theorem 1.1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a harmonically convex function on an interval I and $f \in L[a, b]$, where $a, b \in I$ with $a < b$.*

$$f\Big(\frac{2ab}{a+b}\Big) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

Also, in [2, 3], Imdat Işcan established some new Hermite-Hadamard type inequalities, which estimate the difference between the middle and the right-most terms in (4), for harmonically convex functions:

Definition 3. Let I be an interval in $R_+ \setminus \{0\}$. A function $f : I \to R$ is said to be harmonically quasi-convex on I if the inequality

$$f\Big(\frac{xy}{tx + (1-t)y}\Big) \leq \sup\left\{f\left(x\right), f\left(y\right)\right\} \tag{5}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If this inequality is reversed, then f is said to be harmonically quasi-concave.

Definition 4. The hypergeometric function ${}_2F_1[a,b,c,x]$ is defined for $\mid x \mid < 1$ by the power series

$$ {}_2F_1[a,b,c,x] = \Sigma_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. $$

It is undefined if c equals a non-positive integer. Here $(q)_n$ is the Pochhammer symbol, which is defined by

$$ (q)_n = \begin{cases} 1, & n = 0 \\ q(q+1)\cdots(q+n-1), & n > 0. \end{cases} $$

Definition 5. The beta function, also called the Euler integral of the first kind, is a special function defined by

$$ \beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t. $$

In this paper, we give some generalized inequalities connected with the generalized Hermite-Hadamard-like type inequalities for functions whose derivatives in the absolute value are differentiable harmonically quasi-convex.

2 Main results

In order to find some new inequalities of Hermite-Hadamard-like type inequalities for functions whose derivatives are harmonically quasi-convex, we need the following lemma:

Lemma 1. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L([a,b])$, where $a,b \in I$ with $a < b$ and $\alpha, \lambda \in [0,1]$. Then the following identity*

$$ \begin{aligned} & I(f;a,b;\alpha,\lambda) \\ & \equiv (1-\lambda)\Big\{\frac{(1-\alpha)af(b)+\alpha bf(c)}{c}\Big\} \\ & \qquad + \lambda f\Big\{\frac{(1-\alpha)af(c)+\alpha bf(a)}{c}\Big\} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\ & = ab(b-a)\Big[\alpha^2 \int_0^1 \frac{t-\lambda}{\{ta+(1-t)c\}^2} f'\Big(\frac{ac}{ta+(1-t)c}\Big)\mathrm{d}t \\ & \qquad - (1-\alpha)^2 \int_0^1 \frac{t-1+\lambda}{\{tb+(1-t)c\}^2} f'\Big(\frac{bc}{tb+(1-t)c}\Big)\mathrm{d}t\Big] \end{aligned} \tag{6} $$

holds, for $t \in [0,1]$, where $c = (1-\alpha)a + \alpha b$.

Proof. Firstly, suppose that $\alpha \in (0,1)$ and let $c = (1-\alpha)a+\alpha b$. Integrating by parts, we have:

$$\begin{aligned}
&\int_0^1 \frac{t-\lambda}{\{ta+(1-t)c\}^2} f'\Big(\frac{ac}{ta+(1-t)c}\Big)\mathrm{d}t \\
&= \frac{1}{\alpha a(b-a)c}\int_0^1 (t-\lambda)\mathrm{d}f\Big(\frac{ac}{ta+(1-t)c}\Big)\mathrm{d}t \\
&= \frac{1}{\alpha a(b-a)c}\Big[(1-\lambda)f(c)+\lambda f(a)-\frac{ac}{\alpha(b-a)}\int_a^c \frac{f(x)}{x^2}\mathrm{d}x\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \frac{t-1+\lambda}{\{tb+(1-t)c\}^2} f'\Big(\frac{bc}{tb+(1-t)c}\Big)\mathrm{d}t \\
&= -\frac{1}{(1-\alpha)b(b-a)c}\int_0^1 (t-1+\lambda)\mathrm{d}f\Big(\frac{bc}{tb+(1-t)c}\Big)\mathrm{d}t \\
&= -\frac{1}{(1-\alpha)b(b-a)c} \\
&\quad \times \Big[(1-\lambda)f(b)+\lambda f(c)-\frac{bc}{(1-\alpha)(b-a)}\int_c^b \frac{f(x)}{x^2}\mathrm{d}x\Big].
\end{aligned}$$

By the simple calculations, this case is proved. Secondly, suppose that $\alpha \in \{0,1\}$. The equalities

$$\begin{aligned}
&(i)(1-\lambda)f(b)+\lambda f(a)-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\
&\quad = -ab(b-a)\int_0^1 \frac{t-1+\lambda}{((1-t)a+tb)^2} f'\Big(\frac{ab}{(1-t)a+tb}\Big)\mathrm{d}t \\
&(ii)(1-\lambda)f(b)+\lambda f(a)-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \\
&\quad = ab(b-a)\int_0^1 \frac{t-\lambda}{((1-t)b+ta)^2} f'\Big(\frac{ab}{(1-t)b+ta}\Big)\mathrm{d}t
\end{aligned}$$

can be proved by performing an integration by parts in the integrals from the right side and changing the variable.

Now we turn our attention to establish some inequalities of Hermit-Hadamard-like type for differentiable harmonically quasi-convex functions.

Theorem 2.1. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a,b \in I$ with $a < b$ and $\alpha,\lambda \in [0,1]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \geq 1$ with*

$\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\le ab(b-a)\Big[\alpha^2\mu_1^{\frac{1}{q}}(a,c;\alpha,\lambda;q)\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}} \\
&\quad+(1-\alpha)^2\mu_1^{\frac{1}{q}}(b,c;\alpha,1-\lambda;q)\Big\{\sup\big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

where

$$\begin{aligned}
&\mu_1(a,b;\alpha,\lambda;q) \\
&\quad=\frac{\lambda^{1+q}+(-\lambda)^{1+q}}{c^{2q}(1+q)}{}_2F_1[1,2q,2+q,\frac{\lambda(c-a)}{c}] \\
&\qquad+\frac{\{\lambda a+(1-\lambda)c\}^q}{a^{2q-1}c^{2q-1}(c-a)^{1+q}(1-2q)} \\
&\qquad\quad\times\Big\{a^{2q-1}{}_2F_1[1-2q,-q,2-2q,\frac{c}{\lambda a+(1-\lambda)c}] \\
&\qquad\qquad-c^{2q-1}{}_2F_1[1-2q,-q,2-2q,\frac{a}{\lambda a+(1-\lambda)c}]\Big\}.
\end{aligned}$$

Proof. Suppose that $q\ge 1$ and $c=(1-\alpha)a+\alpha b$. From Lemma 1 and using the power mean integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\le ab(b-a)\Big[\alpha^2\int_0^1\frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^2}\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|\mathrm{d}t \\
&\quad+(1-\alpha)^2\int_0^1\frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^2}\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|\mathrm{d}t\Big] \\
&\le ab(b-a)\Big[\alpha^2\Big\{\int_0^1\frac{\big|t-\lambda\big|^q}{\{ta+(1-t)c\}^{2q}}\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad+(1-\alpha)^2\Big\{\int_0^1\frac{\big|t-1+\lambda\big|^q}{\{tb+(1-t)c\}^{2q}}\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big].
\end{aligned}\tag{7}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q\ge 1$, we know that for $t\in[0,1]$

$$\Big|f'\Big(\frac{ab}{tb+(1-t)a}\Big)\Big|^q\le\sup\Big\{\Big|f'(a)\Big|^q,\Big|f'(b)\Big|^q\Big\},$$

hence, by simple calculation, we have that:

$$
\begin{aligned}
(i) &\int_0^1 \frac{\left|t-\lambda\right|^q}{\{ta+(1-t)c\}^{2q}}\left|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\right|^q \mathrm{d}t \\
&\le \Big(\int_0^1 \frac{\left|t-\lambda\right|^q}{\{ta+(1-t)c\}^{2q}}\mathrm{d}t\Big)\Big(\sup\Big\{\left|f'(a)\right|^q, \left|f'(b)\right|^q\Big\}\Big) \\
&= \mu_1(a,c;\alpha,\lambda;q)\Big(\sup\Big\{\left|f'(a)\right|^q, \left|f'(c)\right|^q\Big\}\Big), \qquad (8)\\
(ii) &\int_0^1 \frac{\left|t-1+\lambda\right|^q}{\{tb+(1-t)c\}^{2q}}\left|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\right|^q \mathrm{d}t \\
&\le \mu_1(b,c;\alpha,1-\lambda;q)\Big(\sup\Big\{\left|f'(b)\right|^q, \left|f'(c)\right|^q\Big\}\Big), \qquad (9)
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
&\int_0^1 \frac{\left|t-\lambda\right|^q}{\{ta+(1-t)c\}^{2q}}\mathrm{d}t = \mu_1(a,c;\alpha,\lambda;q),\\
&\int_0^1 \frac{\left|t-1+\lambda\right|^q}{\{tb+(1-t)c\}^{2q}}\mathrm{d}t = \mu_1(b,c;\alpha,1-\lambda;q).
\end{aligned}
$$

By substituting (8) and (9) in (7), we get the desired result.

Theorem 2.2. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a,b \in I$ with $a<b$ and $\alpha,\lambda \in [0,1]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$
\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\le ab(b-a)\Big(\lambda^2-\lambda+\frac{1}{2}\Big)^{\frac{1}{p}} \\
&\quad\times\Big[\alpha^2\mu_2^{\frac{1}{q}}(a,c;\lambda;q)\Big\{\sup\big\{\left|f'(a)\right|^q, \left|f'(c)\right|^q\big\}\Big\}^{\frac{1}{q}} \\
&\quad+(1-\alpha)^2\mu_2^{\frac{1}{q}}(b,c;1-\lambda;q)\Big\{\sup\big\{\left|f'(b)\right|^q, \left|f'(c)\right|^q\big\}\Big\}^{\frac{1}{q}}\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
&\mu_2(a,c;\lambda;q) \\
&\quad= \frac{1}{2(2q-1)(q-1)(c-a)^2}\Big\{2\{\lambda a+(1-\lambda)c\}^{2-2q} \\
&\quad+c^{1-2q}\{2\lambda(q-1)a+(-1+2\lambda-2\lambda q)c\} \\
&\quad+a^{1-2q}\{(1-2\lambda+2\lambda q)a+2(-1+\lambda+q-\lambda q)c\}\Big\}.
\end{aligned}
$$

Proof. Suppose that $q \geq 1$ and $c = (1-\alpha)a + \alpha b$. From Lemma 1 and using the power mean integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\leq ab(b-a)\Big[\alpha^2\Big\{\int_0^1 \big|t-\lambda\big|\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times \Big\{\int_0^1 \frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^{2q}}\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q \mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2\Big\{\int_0^1 \big|t-1+\lambda\big|\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times \Big\{\int_0^1 \frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^{2q}}\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q \mathrm{d}t\Big\}^{\frac{1}{q}}\Big]. \qquad (10)
\end{aligned}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \geq 1$, we know that for $t \in [0,1]$

$$\Big|f'\Big(\frac{ab}{tb+(1-t)a}\Big)\Big|^q \leq \sup\Big\{\Big|f'(a)\Big|^q, \Big|f'(b)\Big|^q\Big\},$$

hence, by simple calculation, we have that:

$$\begin{aligned}
(i) &\int_0^1 \frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^{2q}}\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q \mathrm{d}t \\
&\leq \Big(\int_0^1 \frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^{2q}}\mathrm{d}t\Big)\Big(\sup\Big\{\big|f'(a)\big|^q, \big|f'(c)\big|^q\Big\}\Big) \\
&= \mu_2(a,c;\lambda;q)\Big\{\sup\big\{\big|f'(a)\big|^q, \big|f'(c)\big|^q\big\}\Big\}, \qquad (11) \\
(ii) &\int_0^1 \frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^{2q}}\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q \mathrm{d}t \\
&\leq \Big(\int_0^1 \frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^{2q}}\mathrm{d}t\Big)\Big(\sup\Big\{\big|f'(b)\big|^q, \big|f'(c)\big|^q\Big\}\Big) \\
&= \mu_2(b,c;1-\lambda;q)\Big\{\sup\big\{\big|f'(b)\big|^q, \big|f'(c)\big|^q\big\}\Big\}, \qquad (12)
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\int_0^1 \frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^{2q}}\mathrm{d}t &= \mu_2(a,c;\lambda;q), \\
\int_0^1 \frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^{2q}}\mathrm{d}t &= \mu_2(b,c;1-\lambda;q).
\end{aligned}$$

Note that

$$\int_0^1 \big|t-\lambda\big|\mathrm{d}t = \int_0^1 \big|t-\lambda\big|\mathrm{d}t = \lambda^2 - \lambda + \frac{1}{2}. \qquad (13)$$

By substituting (11)-(13) in (10), we get the desired result.

Theorem 2.3. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\leq ab(b-a)\Big[\alpha^2\mu_3(a,c;\lambda)\Big\{\sup\big\{\big|f'(a)\big|^q, \big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2\mu_3(b,c;1-\lambda)\Big\{\sup\big\{\big|f'(b)\big|^q, \big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

where

$$\mu_3(a,c;\lambda) = \frac{(1-\lambda)c - \lambda a}{ac(c-a)} + \ln\Big[\frac{ac}{\{\lambda a + (1-\lambda)c\}^2}\Big].$$

Proof. From Lemma 1 and using the power mean integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\leq ab(b-a)\Big[\alpha^2\Big\{\int_0^1 \frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^2}\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times \Big\{\int_0^1 \frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^2}\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2\Big\{\int_0^1 \frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^2}\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times \Big\{\int_0^1 \frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^2}\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big]. \qquad (14)
\end{aligned}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \geq 1$, we have that:

$$\begin{aligned}
(i)\ &\int_0^1 \frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^2}\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q\mathrm{d}t \\
&\leq \Big(\int_0^1 \frac{\big|t-\lambda\big|}{\{ta+(1-t)c\}^2}\mathrm{d}t\Big)\Big(\sup\big\{\big|f'(a)\big|^q, \big|f'(c)\big|^q\big\}\Big) \\
&= \mu_3(a,c;\lambda)\Big\{\sup\big\{\big|f'(a)\big|^q, \big|f'(c)\big|^q\big\}\Big\}, \qquad (15) \\
(ii)\ &\int_0^1 \frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^2}\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q\mathrm{d}t \\
&\leq \Big(\int_0^1 \frac{\big|t-1+\lambda\big|}{\{tb+(1-t)c\}^2}\mathrm{d}t\Big)\Big(\sup\big\{\big|f'(b)\big|^q, \big|f'(c)\big|^q\big\}\Big) \\
&= \mu_3(b,c;1-\lambda)\Big\{\sup\big\{\big|f'(b)\big|^q, \big|f'(c)\big|^q\big\}\Big\}, \qquad (16)
\end{aligned}$$

where we have used the fact that

$$\int_0^1 \frac{\left|t-\lambda\right|}{\{ta+(1-t)c\}^2}\mathrm{d}t = \mu_3(a,c;\lambda),$$
$$\int_0^1 \frac{\left|t-1+\lambda\right|}{\{tb+(1-t)c\}^2}\mathrm{d}t = \mu_3(b,c;1-\lambda).$$

By substituting (15) and (16) in (14), we get the desired result.

Corollary 2.1. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a,b \in I$ with $a < b$ and $\alpha,\lambda \in [0,1]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\Big|I_f(a,b;\alpha,\lambda)\Big| \leq ab(b-a)\min\big\{\omega_1,\omega_2,\omega_3\big\},$$

where

$$\begin{aligned}
\omega_1 &= \alpha^2\mu_1^{\frac{1}{q}}(a,c;\alpha,\lambda;q)\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\\
&\quad+(1-\alpha)^2\mu_1^{\frac{1}{q}}(b,c;\alpha,1-\lambda;q)\Big\{\sup\big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}},\\
\omega_2 &= \Big(\lambda^2-\lambda+\frac{1}{2}\Big)^{\frac{1}{p}}\Big[\alpha^2\mu_2(a,c;\lambda;q)\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\\
&\quad+(1-\alpha)^2\mu_2(b,c;1-\lambda;q)\Big\{\sup\big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\Big],\\
\omega_3 &= \alpha^2\mu_3(a,c;\lambda)\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\\
&\quad+(1-\alpha)^2\mu_3(b,c;1-\lambda)\Big\{\sup\big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.4. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a,b \in I$ with $a < b$ and $\alpha,\lambda \in [0,1]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big|\\
&\leq ab(b-a)M_4^{\frac{1}{p}}(\lambda,p)\Big[\alpha^2\mu_4^{\frac{1}{q}}(a,c;q)\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\\
&\quad+(1-\alpha)^2\mu_4^{\frac{1}{q}}(b,c;q)\Big\{\sup\big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

where

$$M_4(\lambda,p) = \frac{\lambda^{1+p}+(1-\lambda)^{1+p}}{1+p},$$
$$\mu_4(a,c;q) = \frac{c^{1-2q}-a^{1-2q}}{(1-2q)(c-a)}.$$

Proof. From Lemma 1 and using the Hölder's integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\le ab(b-a)\Big[\alpha^2\Big\{\int_0^1 \Big|t-\lambda\Big|^p \mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times \Big\{\int_0^1 \frac{1}{\{ta+(1-t)c\}^{2q}}\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q \mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2\Big\{\int_0^1 \Big|t-1+\lambda\Big|^p \mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times \Big\{\int_0^1 \frac{1}{\{tb+(1-t)c\}^{2q}}\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q \mathrm{d}t\Big\}^{\frac{1}{q}}\Big]. \qquad (17)
\end{aligned}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q>1$, we have that:

$$\begin{aligned}
(i) &\int_0^1 \frac{1}{\{ta+(1-t)c\}^{2q}}\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q \mathrm{d}t \\
&\le \Big(\int_0^1 \frac{1}{\{ta+(1-t)c\}^{2q}}\mathrm{d}t\Big)\Big\{\sup\big\{\big|f'(a)\big|^q, \big|f'(c)\big|^q\big\}\Big\} \\
&= \mu_4(a,c;q)\Big\{\sup\big\{\big|f'(a)\big|^q, \big|f'(c)\big|^q\big\}\Big\}, \qquad (18) \\
(ii) &\int_0^1 \frac{1}{\{tb+(1-t)c\}^{2q}}\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q \mathrm{d}t \\
&\le \Big(\int_0^1 \frac{1}{\{tb+(1-t)c\}^{2q}}\mathrm{d}t\Big)\Big\{\sup\big\{\big|f'(b)\big|^q, \big|f'(c)\big|^q\big\}\Big\} \\
&= \mu_4(b,c;q)\Big\{\sup\big\{\big|f'(b)\big|^q, \big|f'(c)\big|^q\big\}\Big\}. \qquad (19)
\end{aligned}$$

Note that

$$\int_0^1 \Big|t-\lambda\Big|^p \mathrm{d}t = \int_0^1 \Big|t-1+\lambda\Big|^p \mathrm{d}t = M_4(\lambda,p). \qquad (20)$$

and

$$\begin{aligned}
&\int_0^1 \frac{1}{\{ta+(1-t)c\}^{2q}}\mathrm{d}t = \mu_4(a,c;q), \\
&\int_0^1 \frac{1}{\{tb+(1-t)c\}^{2q}}\mathrm{d}t = \mu_4(b,c;q). \qquad (21)
\end{aligned}$$

By substituting (18)-(21) in (17), we get the desired result.

Theorem 2.5. *Let $f: I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a,b])$, where $a,b \in I$ with $a<b$*

and $\alpha, \lambda \in [0,1]$. *If* $|f'|^q$ *is harmonically quasi-convex on* $[a,b]$ *for* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality holds:*

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\leq ab(b-a)\Big[\alpha^2\mu_1^{\frac{1}{p}}(a,c;\alpha,\lambda;p)\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2\mu_1^{\frac{1}{p}}(b,c;\alpha,1-\lambda;p)\Big\{\sup\big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

where μ_1 *is defined as in Theorem 2.1.*

Proof. From Lemma 1 and using Hölder integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\leq ab(b-a)\Big[\alpha^2\Big\{\int_0^1 \frac{\big|t-\lambda\big|^p}{\{ta+(1-t)c\}^{2p}}\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times\Big\{\int_0^1 \Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2\Big\{\int_0^1 \frac{\big|t-1+\lambda\big|^p}{\{tb+(1-t)c\}^{2p}}\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times\Big\{\int_0^1 \Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big].
\end{aligned} \tag{22}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$, we have that:

$$(i)\int_0^1 \Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q\mathrm{d}t \leq \sup\Big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\Big\}, \tag{23}$$

$$(ii)\int_0^1 \Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q\mathrm{d}t \leq \sup\Big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\Big\}. \tag{24}$$

Note that

$$\begin{aligned}
&\int_0^1 \frac{\big|t-\lambda\big|^p}{\{ta+(1-t)c\}^{2p}}\mathrm{d}t = \mu_1(a,c;\alpha,\lambda;p), \\
&\int_0^1 \frac{\big|t-1+\lambda\big|^p}{\{tb+(1-t)c\}^{2p}}\mathrm{d}t = \mu_1(b,c;\alpha,1-\lambda;p).
\end{aligned} \tag{25}$$

By substituting (23)-(25) in (22), we get the desired result.

Theorem 2.6. *Let* $f : I \subseteq R_+ = (0,\infty) \to R$ *be a differentiable function on the interior* I^0 *of an interval* I *and* $f' \in L([a,b])$, *where* $a, b \in I$ *with* $a < b$

and $\alpha, \lambda \in [0,1]$. *If* $|f'|^q$ *is harmonically quasi-convex on* $[a,b]$ *for* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality holds:*

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\leq ab(b-a)M_4^{\frac{1}{q}}(\lambda,q)\Big[\alpha^2\mu_4^{\frac{1}{p}}(a,c;p)\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2\mu_4^{\frac{1}{p}}(b,c;p)\Big\{\sup\big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\big\}\Big\}^{\frac{1}{q}}\Big],
\end{aligned}$$

where M_4 *and* μ_4 *is defined as in Theorem 2.4.*

Proof. From Lemma 1 and using Hölder integral inequality, we have

$$\begin{aligned}
&\Big|I_f(a,b;\alpha,\lambda)\Big| \\
&\leq ab(b-a)\Big[\alpha^2\Big\{\int_0^1 \frac{1}{\{ta+(1-t)c\}^{2p}}\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times\Big\{\int_0^1 \big|t-\lambda\big|^q\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2\Big\{\int_0^1 \frac{1}{\{tb+(1-t)c\}^{2p}}\mathrm{d}t\Big\}^{\frac{1}{p}} \\
&\qquad \times\Big\{\int_0^1 \big|t-1+\lambda\big|^q\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q\mathrm{d}t\Big\}^{\frac{1}{q}}\Big].
\end{aligned} \tag{26}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$, we have that:

$$\begin{aligned}
(i)\int_0^1 &\Big|t-\lambda\Big|^q\Big|f'\Big(\frac{ac}{ta+(1-t)c}\Big)\Big|^q\mathrm{d}t \\
&\leq \Big\{\int_0^1 \Big|t-\lambda\Big|^q\mathrm{d}t\Big\}\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}
\end{aligned}$$

$$= M_4(\lambda,q)\Big\{\sup\big\{\big|f'(a)\big|^q,\big|f'(c)\big|^q\big\}\Big\}, \tag{27}$$

$$\begin{aligned}
(ii)\int_0^1 &\Big|t-1+\lambda\Big|^q\Big|f'\Big(\frac{bc}{tb+(1-t)c}\Big)\Big|^q\mathrm{d}t \\
&\leq M_4(\lambda,q)\Big\{\sup\big\{\big|f'(b)\big|^q,\big|f'(c)\big|^q\big\}\Big\}.
\end{aligned} \tag{28}$$

Note that

$$\begin{aligned}
\int_0^1 \frac{1}{\{ta+(1-t)c\}^{2p}}\mathrm{d}t &= \mu_4(a,c;p), \\
\int_0^1 \frac{1}{\{tb+(1-t)c\}^{2p}}\mathrm{d}t &= \mu_4(b,c;p).
\end{aligned} \tag{29}$$

By substituting (27)-(29) in (26), we get the desired result.

Corollary 2.2. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\Big| I_f(a, b; \alpha, \lambda) \Big| \leq ab(b-a) \min \big\{ \omega_4, \omega_5, \omega_6 \big\},$$

where

$$\begin{aligned}
\omega_4 &= M_4^{\frac{1}{p}}(\lambda, p) \Big[\alpha^2 \mu_4^{\frac{1}{q}}(a, c; q) \Big\{ \sup \big\{ \big|f'(a)\big|^q, \big|f'(c)\big|^q \big\} \Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2 \mu_4^{\frac{1}{q}}(b, c; q) \Big\{ \sup \big\{ \big|f'(b)\big|^q, \big|f'(c)\big|^q \big\} \Big\}^{\frac{1}{q}} \Big], \\
\omega_5 &= \alpha^2 \mu_1^{\frac{1}{p}}(a, c; \alpha, \lambda; p) \Big\{ \sup \big\{ \big|f'(a)\big|^q, \big|f'(c)\big|^q \big\} \Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2 \mu_1^{\frac{1}{p}}(b, c; \alpha, 1-\lambda; p) \Big\{ \sup \big\{ \big|f'(b)\big|^q, \big|f'(c)\big|^q \big\} \Big\}^{\frac{1}{q}}, \\
\omega_6 &= M_4^{\frac{1}{q}}(\lambda, q) \Big[\alpha^2 \mu_4^{\frac{1}{p}}(a, c; p) \Big\{ \sup \big\{ \big|f'(a)\big|^q, \big|f'(c)\big|^q \big\} \Big\}^{\frac{1}{q}} \\
&\quad + (1-\alpha)^2 \mu_4^{\frac{1}{p}}(b, c; p) \Big\{ \sup \big\{ \big|f'(b)\big|^q, \big|f'(c)\big|^q \big\} \Big\}^{\frac{1}{q}} \Big].
\end{aligned}$$

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Received: July 3, 2014