Symmetry Identities for the Generalized
Higher-Order $q$-Bernoulli Polynomials
under $S_3$

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Abstract

In this paper, we study the identities of symmetry for the generalized higher-order \( q \)-Bernoulli polynomials under \( S_3 \).

1. Introduction

Let \( p \) be a fixed prime number. Throughout this paper, \( \mathbb{Z}_p \), \( \mathbb{Q}_p \) and \( \mathbb{C}_p \) denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. The \( p \)-adic norm \( | \cdot |_p \) is normalized as \( | \cdot |_p = 1/p \). Let \( UD(\mathbb{Z}_p) \) be the space of all uniformly differentiable functions on \( \mathbb{Z}_p \) and the \( q \)-number of \( x \) is defined by \( [x]_q = (1 - q^x)/(1 - q) \). Note that \( \lim_{q \to 1} [x]_q = x \). For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim to be

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad \text{(see [12, 13])}. \tag{1}
\]

For \( d \in \mathbb{N} \) with \((d, p) = 1\), we set \( \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z} \rightarrow X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} (a + dp\mathbb{Z}_p) \)

and

\[
a + dp^N\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},
\]

where \( a \in \mathbb{Z} \) lies \( 0 \leq a < dp^N \).

Let \( \chi \) be a primitive Dirichlet character with conductor \( d \in \mathbb{N} \). Then the generalized Carlitz’s \( q \)-Bernoulli polynomials attached to \( \chi \) are given by Kim to be

\[
\beta_{n, \chi, q}(x) = \int_X \chi(y)[x + y]_q^n d\mu_q(y), \quad (n \geq 0), \quad \text{(see [1 – 20])}. \tag{2}
\]

When \( x = 0 \), \( \beta_{n, \chi, q} = \beta_{n, \chi, q}(0) \) are called the generalized Carlitz \( q \)-Bernoulli numbers, (see [1-20]).

In this paper, we consider the generalized higher-order \( q \)-Bernoulli polynomials attached to \( \chi \) and give symmetric identities of those polynomials in three variables under \( S_3 \).

2. Symmetry identities of generalized higher-order \( q \)-Bernoulli polynomials

For \( r \in \mathbb{N} \), let us consider the generalized higher-order \( q \)-Bernoulli polynomials attached to \( \chi \) as follows:

\[
\int_X \cdots \int_X \prod_{\ell=1}^{r} (\chi(x_\ell)) e^{\sum_{i=1}^{r} x_\ell + x_\ell^t} d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \beta_{n, \chi, q}^{(r)}(x) \frac{t^n}{n!}. \tag{3}
\]
Symmetry identities for generalized $q$-Bernoulli polynomials

By (3), we get

$$
\int_X \cdots \int_X \prod_{\ell=1}^r (\chi(x_\ell))[x_1 + \cdots + x_r + x]^n q \, d\mu_q(x_1) \cdots d\mu_q(x_r) = \beta^{(r)}_{n,\chi,q}(x), \quad (n \geq 0).
$$

(4)

When $x = 0$, $\beta^{(r)}_{n,\chi,q} = \beta^{(r)}_{n,\chi,q}(0)$ are called the generalized higher-order $q$-Bernoulli numbers attached to $\chi$.

Let $w_1, w_2, w_3 \in \mathbb{N}$. Then, we have

$$
\int_X \cdots \int_X \prod_{\ell=1}^r (\chi(x_\ell))
\times e^{[w_2 w_3 \sum_{\ell=1}^r x_\ell + w_1 w_2 w_3 x + w_1 w_3 \sum_{\ell=1}^r i_\ell + w_1 w_2 \sum_{\ell=1}^r j_\ell] q t} \, d\mu_{q w_2 w_3}(x_1) \cdots d\mu_{q w_2 w_3}(x_r)

= \lim_{N \to \infty} \left( \frac{1}{[w_1 dq^N]_{q w_2 w_3}} \right)^r \sum_{x_1, \ldots, x_r = 0}^{dN-1} \left( \prod_{\ell=1}^r \chi(x_\ell) \right) q^{w_2 w_3 \sum_{\ell=1}^r x_\ell + w_1 w_2 w_3 x + w_1 w_3 \sum_{\ell=1}^r i_\ell + w_1 w_2 \sum_{\ell=1}^r j_\ell} t

\times e^{[w_2 w_3 \sum_{\ell=1}^r (k_\ell + w_1 x_\ell) + w_1 w_2 w_3 x + w_1 w_3 \sum_{\ell=1}^r i_\ell + w_1 w_2 \sum_{\ell=1}^r j_\ell] q t}.
$$

(5)

From (5), we have

$$
\left( \frac{1}{[w_2 w_3]_q} \right)^r \sum_{i_1, \ldots, i_r = 0}^{dN-1} \sum_{j_1, \ldots, j_r = 0}^{dN-1} \left( \prod_{\ell=1}^r \chi(i_\ell) \chi(j_\ell) \right) q^{w_1 w_3 \sum_{\ell=1}^r i_\ell + w_1 w_2 \sum_{\ell=1}^r j_\ell} t

\times \int_X \cdots \int_X \left( \prod_{\ell=1}^r (\chi(x_\ell)) \right) e^{[w_2 w_3 \sum_{\ell=1}^r x_\ell + w_1 w_2 w_3 x + w_1 w_3 \sum_{\ell=1}^r i_\ell + w_1 w_2 \sum_{\ell=1}^r j_\ell] q t} \, d\mu_{q w_2 w_3}(x_1) \cdots d\mu_{q w_2 w_3}(x_r)

= \lim_{N \to \infty} \left( \frac{1}{[dw_1 w_2 w_3 p^N]_q} \right)^r \sum_{i_1, \ldots, i_r = 0}^{dwN-1} \sum_{j_1, \ldots, j_r = 0}^{dwN-1} \sum_{k_1, \ldots, k_r = 0}^{dpN-1} \left( \prod_{\ell=1}^r \chi(i_\ell) \chi(j_\ell) \chi(k_\ell) \right)

\times q^{w_1 w_3 \sum_{\ell=1}^r i_\ell + w_1 w_2 \sum_{\ell=1}^r j_\ell + w_1 w_3 \sum_{\ell=1}^r k_\ell} \, d\mu_{q w_1 w_2 w_3} \sum_{\ell=1}^N x_\ell

\times e^{[w_2 w_3 \sum_{\ell=1}^r (k_\ell + w_1 x_\ell) + w_1 w_2 w_3 x + w_1 w_3 \sum_{\ell=1}^r i_\ell + w_1 w_2 \sum_{\ell=1}^r j_\ell] q t}.
$$

(6)

As this expression is invariant under any permutation $\sigma \in S_3$, we have the following theorem.
Theorem 2.1. For $d, w_1, w_2, w_3 \in \mathbb{N}$, the following expressions

$$
\left( \frac{1}{w_{\sigma(2)} w_{\sigma(3)}} \right)_q \sum_{i_1 \ldots i_r = 0}^{r} \sum_{j_1 \ldots j_r = 0}^{r} \left( \prod_{\ell=1}^{r} \chi(i_\ell) \chi(j_\ell) \right)
\times q^{w_{\sigma(1)} w_{\sigma(3)} \sum_{\ell=1}^{r} i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^{r} j_\ell} \int_{X} \ldots \int_{X} \left( \prod_{\ell=1}^{r} \chi(x_\ell) \right)
\times d\mu_{q^{w_{\sigma(2)} w_{\sigma(3)}}} (x_1) \cdots d\mu_{q^{w_{\sigma(2)} w_{\sigma(3)}}} (x_r)
$$

are the same for any permutation $\sigma \in S_3$.

From (4), we can derive the following equation:

$$
\int_{X} \ldots \int_{X} \left( \prod_{\ell=1}^{r} \chi(x_\ell) \right) e^{[w_{\sigma(2)} w_{\sigma(3)} \sum_{\ell=1}^{r} i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^{r} j_\ell] q} \int_{X} \ldots \int_{X} \left( \prod_{\ell=1}^{r} \chi(x_\ell) \right)
\times d\mu_{q^{w_{\sigma(2)} w_{\sigma(3)}}} (x_1) \cdots d\mu_{q^{w_{\sigma(2)} w_{\sigma(3)}}} (x_r)
\times \sum_{n=0}^{\infty} \frac{n!}{n^{n}}
$$

Therefore, by (7) and Theorem 1, we obtain the following corollary.

Corollary 2.2. For $n \in \mathbb{N} \cup \{0\}, d, w_1, w_2, w_3 \in \mathbb{N}$, the following expressions

$$
\left( \frac{1}{w_{\sigma(2)} w_{\sigma(3)}} \right)_q \sum_{i_1 \ldots i_r = 0}^{r} \sum_{j_1 \ldots j_r = 0}^{r} \left( \prod_{\ell=1}^{r} \chi(i_\ell) \chi(j_\ell) \right)
\times q^{w_{\sigma(1)} w_{\sigma(3)} \sum_{\ell=1}^{r} i_\ell + w_{\sigma(1)} w_{\sigma(2)} \sum_{\ell=1}^{r} j_\ell} \beta^{(r)}_{n, \chi, q^{w_{\sigma(2)} w_{\sigma(3)}}} (w_{\sigma(1)} x + w_{\sigma(2)} \sum_{\ell=1}^{r} i_\ell + w_{\sigma(3)} \sum_{\ell=1}^{r} j_\ell)
$$

are the same for any $\sigma \in S_3$. 


By (4), we get
\[
\int_X \cdots \int_X \left( \prod_{\ell=1}^{r} \chi(x_{\ell}) \right) \left[ \sum_{\ell=1}^{r} x_{\ell} + w_1 x + w_1 \sum_{\ell=1}^{r} i_{\ell} + w_1 \sum_{\ell=1}^{r} j_{\ell} \right]^{n} \times d\mu_{q^{w_2}w_3}(x_1) \cdots d\mu_{q^{w_2}w_3}(x_r)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{[w_1]}{[w_2w_3]} \right)^{n-k} \left[ w_3 \sum_{\ell=1}^{r} i_{\ell} + w_2 \sum_{\ell=1}^{r} j_{\ell} \right]^{n-k} q^{k(w_1 w_3 \sum_{\ell=1}^{r} i_{\ell} + w_1 w_2 \sum_{\ell=1}^{r} j_{\ell})} q^{w_1} \times \int_X \cdots \int_X \left( \prod_{\ell=1}^{r} \chi(x_{\ell}) \right) \left[ \sum_{\ell=1}^{r} x_{\ell} + w_1 x + w_1 \sum_{\ell=1}^{r} i_{\ell} + w_1 \sum_{\ell=1}^{r} j_{\ell} \right]^{n} \times d\mu_{q^{w_2}w_3}(x_1) \cdots d\mu_{q^{w_2}w_3}(x_r)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{[w_1]}{[w_2w_3]} \right)^{n-k} \left[ w_3 \sum_{\ell=1}^{r} i_{\ell} + w_2 \sum_{\ell=1}^{r} j_{\ell} \right]^{n-k} q^{k(w_1 w_3 \sum_{\ell=1}^{r} i_{\ell} + w_1 w_2 \sum_{\ell=1}^{r} j_{\ell})} q^{w_1} \times \beta_{k,r}^{(r)}(w_1 x).
\]

From (7) and (8), we have
\[
\frac{[w_2w_3]_q^{n}}{[w_2w_3]_q^{r}} \sum_{i_1, \cdots, i_r=0}^{d\mu_{q^{w_2}w_3}} \sum_{j_1, \cdots, j_r=0}^{d\mu_{q^{w_2}w_3}} \left( \prod_{\ell=1}^{r} \chi(i_{\ell})\chi(j_{\ell}) \right) q^{w_1 w_3 \sum_{\ell=1}^{r} i_{\ell} + w_1 w_2 \sum_{\ell=1}^{r} j_{\ell}} q^{w_1} \times \int_X \cdots \int_X \left( \prod_{\ell=1}^{r} \chi(x_{\ell}) \right) \left[ \sum_{\ell=1}^{r} x_{\ell} + w_1 x + w_1 \sum_{\ell=1}^{r} i_{\ell} + w_1 \sum_{\ell=1}^{r} j_{\ell} \right]^{n} \times d\mu_{q^{w_2}w_3}(x_1) \cdots d\mu_{q^{w_2}w_3}(x_r)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \frac{[w_2w_3]_q^{k/r}}{[w_2w_3]_q^{r}} \frac{[w_1]}{[w_2w_3]}_q^{n-k} \beta_{k,r}^{(r)}(w_1 x) S_{n, k, q^{w_2}w_3}(w_2, w_3 : d|\chi),
\]

where \( n \geq 0 \).

Therefore, by (9) and (10), we obtain the following theorem.

**Theorem 2.3.** For \( n \in \mathbb{N} \cup \{0\}, d, w_1, w_2, w_3 \in \mathbb{N} \), the following expressions
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{[w_2]_q^{k/r}}{[w_2]_q} \frac{[w_3]_q^{k}}{[w_3]_q} \beta_{k,r}^{(r)}(w_1^{w_2}w_3^{w_3}, \chi(w_1^{w_2}w_3, \chi(w_1^{w_2}w_3 : d|\chi))}
\]
are the same for any permutation $\sigma \in S_3$.

References


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