Approximations of Best Proximity Points of Cyclic Self-maps in Metric Spaces and the Associated Cyclic Asymptotic Regularity

M. De la Sen

Institute of Research and Development of Processes
University of the Basque Country
Campus of Leioa (Bizkaia) - P.O. Box 644- Bilbao
Barrio Sarriena, 48940- Leioa, Spain

Copyright © 2014 M. De la Sen. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

This manuscript relies on 2-cyclic self-mappings by giving a simple formal development to join some useful concepts, and related results, like those of approximate best proximity points of cyclic self-mappings, the approximate best proximity (respectively, partial best proximity) point property or the cyclic asymptotic regularity.

Keywords: approximate best proximity points, approximate best proximity property, approximate partial best proximity property, cyclic self-maps, cyclic asymptotic regularity.

1. Introduction

Fixed point theory is receiving important attention in the last decades because of its applicability to many problems like stability and stabilization of dynamic systems, convergence of sequences, approximation of solutions of algebraic systems, properties of boundedness, convergence and acceleration of convergence of iterative schemes, estimation theory and others. See [1-10] and references therein. Important applications
can and could be got for stability of various types of dynamic systems using approaches based on fixed point tools. See, for instance, [11-13] and references therein. In [4-5] and other cited papers by the same authors and also some references therein, the problem of existence of approximate fixed points of functions in metric spaces is described and formalized as well as its links with asymptotic regularity of such functions including cases where contractive conditions are fulfilled. An important issue of the developed framework is that many of the relevant properties are linked to the existence of nonempty balls around at least one of the points of the set $X$ of the metric space $(X, d)$ being defined by points $x$ and their images through the involved self-mappings, as well as to the asymptotic regularity of such self-mappings. For many of the presented results, special properties of the sets, beyond being nonempty, are not formally required since the formal development is focused on the study of the existence of approximate fixed points. It is not required either that the metric space be complete by the same reasons. This paper extends such a formalism to 2-cyclic self-mappings. In this way, a simple formal development is given which brings together the concepts of approximate best proximity points of 2-cyclic self-mappings, approximate best proximity (respectively, partial best proximity) point property and cyclic asymptotic regularity of 2-cyclic self-mappings. Let us remember that those 2-cyclic self-mappings are defined on the union of two nonempty subsets of the set $X$ where $(X, d)$ is a metric space. Some related properties are proved while certain particular results are derived for contractive cyclic self-mappings $f$ on $A \cup B$ with $A$ and $B$ being nonempty, in general disjoint, subsets of $X$. There are other directly induced results available for the composite self-mapping $f^2$ in $A \cup B$ obtained from the approximate best proximity point properties and those related ones of cyclic asymptotic regularity of $f$ on $A \cup B$. The extensions of all the obtained results to cyclic self-maps defined on any finite union of subsets of $X$ is direct and are not discussed in the paper.

2. Problem statement

Let $(X, d)$ be a metric space and let $f : A \cup B \rightarrow A \cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets $A$ and $B$ of $X$. Since there are only two subsets involved, the self-mapping will be referred to simply as a cyclic self-mapping. The following definition will be then used:

**Definitions 2.1.** Let $(X, d)$ be a metric space and let $f : A \cup B \rightarrow A \cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets $A$ and $B$ of $X$. Then

1. $x \in A \cup B$ is an $\varepsilon$-best proximity point of $f$ (in $A$ or in $B$) for a given $\varepsilon \in R_{0+}$ if $d(x, fx) \leq D + \varepsilon$, where $R_{0+} = R_+ \cup \{0\}$, $R_+ = \{z \in R : z > 0\}$ and $D = d(A, B) = \inf_{x \in A, y \in B} d(x, y)$.

2. $x \in A$ is an $\varepsilon$-best proximity point of $f$ in $A$ for a given $\varepsilon \in R_{0+}$ if $d(x, fx) \leq D + \varepsilon$. $\square$
Approximations of best proximity points

It turns out that \( x \in A \cup B \) is an \( \varepsilon \) - best proximity point of \( f \) if and only if

\[
x \in BP_\varepsilon(f) = \{ x \in A \cup B : d(x, fx) \leq D + \varepsilon \}
\]

Also \( x \in A \) is an \( \varepsilon \) - best proximity point of \( f \) in \( A \) if and only if

\[
x \in BP_{A\varepsilon}(f) = \{ x \in A : d(x, fx) \leq D + \varepsilon \}
\]

The following results are obvious:

**Proposition 2.2.** Let \((X, d)\) be a metric space and let \( f : A \cup B \to A \cup B \) be a 2-cyclic self-mapping on the union of two nonempty subsets \( A \) and \( B \) of \( X \). Then, \( x \in A \cup B \) is an \( \varepsilon \) - best proximity point of \( f \) then it is also an \( \varepsilon_1 \) - best proximity point of \( f \) for any real \( \varepsilon_1 \geq \varepsilon \).

**Proof:** It follows directly since \( BP_\varepsilon(f) \subseteq BP_{\varepsilon_1}(f) \) for any real \( \varepsilon_1 \geq \varepsilon \). \( \square \)

Since all the subsequent developments are given for cyclic self-mappings on the union of two nonempty subsets of \( X \), we will refer 2-cyclic self-mappings simply as cyclic self-mappings for the sake of notational simplicity.

**Proposition 2.3.** Let \((X, d)\) be a metric space and let \( f : A \cup B \to A \cup B \) be a 2-cyclic self-mapping on the union of two bounded nonempty subsets \( A \) and \( B \) of \( X \). Then \( x \in A \cup B \) is an \( \varepsilon \) - best proximity point of \( f \) then \( fx \) is an \( \varepsilon' \) - best proximity point for some \( \varepsilon' \in R_+ \).

**Proof:** Since \( x \in BP_\varepsilon(f) \), \( f^2x \in A \) if \( x \in A \), \( f^2x \in B \) if \( x \in B \), and \( A \) and \( B \) are bounded it follows that

\[
d(fx, f^2x) \leq d(fx, x) + d(x, f^2x) \leq \varepsilon' = D + \varepsilon + \max(diamA, diamB)
\]

Then, \( fx \in BP_{\varepsilon'}(f) \). \( \square \)

**Definitions 2.4.** Let \((X, d)\) be a metric space and let \( f : A \cup B \to A \cup B \) be a 2-cyclic self-mapping on the union of two nonempty subsets \( A \) and \( B \) of \( X \). Then

1. \( f : A \cup B \to A \cup B \) has the approximate best proximity point property if \( BP_\varepsilon(f) \neq \emptyset \) for all \( \varepsilon \in R_+ \).
2. \( f : A \cup B \to A \cup B \) has the approximate best proximity point property in \( A \) if \( BP_{A\varepsilon}(f) \neq \emptyset \) for all \( \varepsilon \in R_+ \).
(3) $f : A \cup B \to A \cup B$ has the $\varepsilon_0$-partial approximate best proximity point property if $BP_{\varepsilon}(f) \neq \emptyset$ for all real $\varepsilon \geq \varepsilon_0$ and a given $\varepsilon_0 \in R_{0+}$.

(4) $f : A \cup B \to A \cup B$ has the $\varepsilon_0$-partial approximate best proximity point property in $A$ if $BP_{\varepsilon}(f) \neq \emptyset$ for all real $\varepsilon \geq \varepsilon_0$ and a given $\varepsilon_0 \in R_{0+}$.

(5) $f^2 : A \cup B \to A \cup B$ has the $\varepsilon_0$-partial approximate fixed point property if $FP_{\varepsilon}(f) = \{x \in A \cup B : d(x, f^2 x) \leq \varepsilon \neq \emptyset ; \forall \varepsilon(\in R_{+}) \}$ for some given $\varepsilon_0 \in R_{0+}$.

(6) $f^2 : A \cup B \to A \cup B$ has the $\varepsilon_0$-partial approximate fixed point property in $A$ if $FP_{\varepsilon}(f) = \{x \in A : d(x, f^2 x) \leq \varepsilon \neq \emptyset ; \forall \varepsilon(\in R_{+}) \}$ for some given $\varepsilon_0 \in R_{0+}$.

(7) $f^2 : A \cup B \to A \cup B$ has the approximate fixed point property if it has the 0-partial approximate fixed point property.

Note that if $A \cap B \neq \emptyset$, and then $D = 0$, then the approximate best proximity point property of $f : A \cup B \to A \cup B$ is equivalent to its approximated fixed point property. Note also that $f : A \cup B \to A \cup B$ has the approximate best proximity point property if it has the 0-partial approximate best proximity point property.

**Definitions 2.5.** Let $(X, d)$ be a metric space and let $A$ and $B$ be nonempty subsets of $X$ with $d(A, B) = D$. Then,

(1) $f : A \cup B \to A \cup B$ is cyclic asymptotically regular if it is cyclic and

$$d(f^n x, f^{n+1} x) \to D \text{ as } n \to \infty ; \forall x \in A \cup B$$

(2) $f : A \cup B \to A \cup B$ is cyclic asymptotically $\varepsilon_0$-regular, respectively, $\varepsilon_0$-regular in $A$, if it is cyclic and

$$d(f^n x, f^{n+1} x) \to D + \varepsilon_0 \text{ as } n \to \infty ; \forall x \in A$$

respectively.

**Proposition 2.6.** Let $(X, d)$ be a metric space and let $A$ and $B$ be nonempty subsets of $X$ with $d(A, B) = D$. Then, any strictly contractive cyclic self-mapping $f : A \cup B \to A \cup B$ is cyclic asymptotically regular, and equivalently, it has the approximate best proximity point property.

**Proof:** It is direct since, if $f : A \cup B \to A \cup B$ then

$$d(f^2 x, f x) \leq Kd(fx, x) + (1 - K)D$$

(2.1)

for some $K \in [0, 1)$; $\forall x \in A \cup B$. Thus, it is cyclic asymptotically regular since
\[ D \leq d(f_{n+1}^x, f^n x) \leq K^nd(x, x) + \left(1 - K^n\right)D \] (2.2)

and
\[ d(f_{n+1}^x, f^n x) \to D \text{ as } n \to \infty ; \quad \forall x \in A \cup B. \] Also, since \( f : A \cup B \to A \cup B \) is cyclic asymptotically regular then there is \( n_0 = n_0(\varepsilon) > 0 \) for any given \( \varepsilon \in \mathbb{R}_+ \) such that
\[ D \leq d(f_{n+1}^x, f^n x) \leq D + \varepsilon \text{ so that } BP_\varepsilon(f) \neq \emptyset ; \quad \forall \varepsilon \in \mathbb{R}_+ , \quad \forall x \in A \cup B. \] As a result, \( f : A \cup B \to A \cup B \) has the approximate best proximity point property. Equivalently, if
\[ BP_\varepsilon(f) \neq \emptyset \Leftrightarrow \left( D \leq d(f_{n+1}^x, f^n x) \leq D + \varepsilon , \quad \forall \varepsilon \in \mathbb{R}_+ , \quad \forall x \in A \cup B , \quad n(\varepsilon) \in \mathbb{Z}_{0+} \right) \]

with \( \mathbb{Z}_{0+} = \{ \varepsilon \in \mathbb{Z} : \varepsilon \geq 0 \} \), then
\[ \lim_{n \to \infty} d(f_{n+1}^x, f^n x) = D ; \quad \forall x \in A \cup B \quad \text{so that } f : A \cup B \to A \cup B \text{ is cyclic asymptotically regular.} \]

\[ \square \]

**Lemma 2.7.** Let \((X, d)\) be a metric space and let \( A \) and \( B \) be nonempty subsets of \( X \) with \( d(A, B) = D \). If \( f : A \cup B \to A \cup B \) is cyclic asymptotically regular then it has the approximate best proximity point property.

**Proof:** For any \( x \in A \cup B \) and any given \( \varepsilon \in \mathbb{R}_+ \), there is \( m = m(\varepsilon) \in \mathbb{Z}_{0+} \) such that
\[ D \leq d(f^n x, f_{n+1} x) \leq D + \varepsilon ; \quad \forall n(\varepsilon) \in \mathbb{Z}_{0+}. \] Thus, \( BP_\varepsilon(f) \neq \emptyset \) for any \( \varepsilon \in \mathbb{R}_+ \) so that \( f : A \cup B \to A \cup B \) has the approximate best proximity point property. \( \square \)

**Lemma 2.8.** Let \((X, d)\) be a metric space and let \( A \) and \( B \) be nonempty bounded subsets of \( X \) with \( d(A, B) = D \). If \( f^2 : A \cup B \to A \cup B \) is asymptotically regular then \( f : A \cup B \to A \cup B \) has the \( \varepsilon_0 \)-partial best proximity point property for some threshold \( \varepsilon_0 \in \mathbb{R}_+ \), with \( \varepsilon_0 \leq \min(\text{diam} A, \text{diam} B) \).

**Proof:** One has
\[ d(f^{2n} x, f^n x) \leq d(f^{2n} x, f^{2n+2} x) + d(f^{2n+2} x, f^{2n+1} x) ; \quad \forall x \in A \cup B \] (2.3)
\[ d(f^{2n+2} x, f^{2n+1} x) \leq d(f^{2n} x, f^{2n+2} x) + d(f^{2n} x, f^{2n+1} x) ; \quad \forall x \in A \cup B \] (2.4)

and, since \( d(f^{2n} x, f^{2n+2} x) \to 0 \) as \( n \to \infty \), \( \forall x \in A \cup B \), since \( f^2 : A \cup B \to A \cup B \) is asymptotically regular, and since \( \{d(f^n x, f^{n+1} x)\} \) is bounded from the boundedness of \( A \) and \( B \) and the fact that \( T(A) \subseteq B \) and \( T(B) \subseteq A \), one has from (2.3)-(2.4):
\[ d(f^{2n} x, f^{2n+1} x) - d(f^{2n+2} x, f^{2n+1} x) \to 0 \text{ as } n \to \infty \text{ so that } d(f^{2n} x, f^{2n+1} x) \to D_0(\geq D) \text{ as } n \to \infty ; \quad \forall x \in A \cup B \]. Thus, \( D_0 \leq D + \min(\text{diam} A, \text{diam} B) < \infty \) since \( A \) and \( B \) are bounded and each \( f^{2n} x, f^{2n+1} x \) is one of them in \( A \) and the other one in \( B \). Then, fix \( x_0 = f^{2n} x \) there is
\[ n_0 \in \mathbb{Z}_{0^+} \text{ such that } d(x_0, f x_0) \leq D + \varepsilon \text{ so that } BP_\varepsilon (f) \neq \emptyset \text{ for any real } min (\text{diam } A, \text{diam } B) \geq \varepsilon \geq \varepsilon_0 \text{ and some } \varepsilon_0 \in \mathbb{R}_{0^+}. \]

Theorem 2.9. Let \((X, d)\) be a complete metric space and let \(A \text{ and } B\) be nonempty closed subsets of \(X\) with \(d(A, B) = D\). Assume that \(A\) is approximatively compact with respect to \(B\). Then \(f : A \cup B \to A \cup B\) is cyclic asymptotically regular if and only if \(f^2 : A \cup B \to A \cup B\) is asymptotically regular.

\textbf{Proof}: Note that, since \(BP_\varepsilon (f) \neq \emptyset \); \(\forall \varepsilon \in \mathbb{R}_{0^+}\), then \(f : A \cup B \to A \cup B\) has 0-best proximity points in \(A\) and in \(B\) for any \(\varepsilon \in \mathbb{R}_{0^+}\) so that, in particular, \(BP_\varepsilon (f) \neq \emptyset\); and \(f : A \cup B \to A \cup B\) has 0-best proximity points. Since \(A\) is approximatively compact with respect to \(B\), the set \(\{ y \in B : d (x, A) = D \}\) is nonempty and, also, \(d(y, x_0) \to d(y, A) = D\) for some \(y \in B\) and some sequence \(\{x_n\} \subset A\), then there is a convergent subsequence \(\{x_{n_k}\} \subset A\) of \(\{x_n\}\).

a) First, it is proved that if \(f : A \cup B \to A \cup B\) is cyclic asymptotically regular then \(f^2 : A \cup B \to A \cup B\) is asymptotically regular. Assume that \(f : A \cup B \to A \cup B\) is cyclic asymptotically regular so that it has the approximate best proximity point property so that \(BP_{\varepsilon} (f) \neq \emptyset\); \(\forall \varepsilon \in \mathbb{R}_{0^+}\), and \(d(f^{n+1}, f^n) \to D\) as \(n \to \infty\); \(\forall x \in A \cup B\). Now, take \(x \in A\). Since \(A\) is approximatively compact with respect to \(B\), then \(\{ y \in B : d (x, A) = D \} \neq \emptyset\), and there is a convergent subsequence in \(A\), \(\{ x_{2n} \} \) of \(\{x_{n_k}\}\) with the properties \(d(x_{n_k}, x_{n+1}) \to D\), \(d(x_{2n}, y) \to D\), \(d(x_{2n}, y) \to D\) as \(n \to \infty\) for \(x \in A\), and \(\{ x_{2n} = f^{2n}(x) \} \to z(\in A)\) with \(x_{n+1} = f x_n = f^{n+1}(x(\in A \cup B), x_{2n} = f^{2n}(x(\in A))\) since \(x \in A\).

Proceed by contradiction by assuming that \(f^2 : A \cup B \to A \cup B\) is not asymptotically regular. Then, there is \(\varepsilon \in \mathbb{R}_{0^+}\) and a sequence of positive integers \(\{n_k\}\) such that \(d(f^{2n_k+1}x, f^{2n_k}x) > \varepsilon\); \(\forall x \in A\) with \(z \in A\) and \(fz \in B\) being best proximity points of \(A\) and \(B\) which are then 0-best proximity points (note that if \(z \in A\) is a 0-best proximity point then \(d(z, fz) = D\) so that \(fz \in B\) is also a 0-best proximity point). Then, the following contradiction follows:

\[ 0 < \varepsilon \leq \liminf_{k \to \infty} f^{2n_k+2}x \leq \liminf_{k \to \infty} f^{2n_k+2}x \leq \lim_{k \to \infty} f^{2n_k+2}x \leq 0 \]

For \(y \in B\) we can repeat all the above reasoning for \(x = f x \in A\).

In conclusion, \(f^2 : A \cup B \to A \cup B\) is asymptotically regular and \(BP_{\varepsilon} (f) \neq \emptyset \Rightarrow [BP_{\varepsilon} (f)] = \{ x \in A \cup B : d(x, f^2 x) \leq \varepsilon \} \neq \emptyset\) for any given \(\varepsilon \in \mathbb{R}_{0^+}\) if \(f : A \cup B \to A \cup B\) is cyclic asymptotically regular.
b) Now, the converse implication is proved, that is, if \( f^2 : A \cup B \to A \cup B \) is asymptotically regular then \( f : A \cup B \to A \cup B \) is cyclic asymptotically regular, equivalently,

\[
[F^2] = \{ x \in A \cup B : d(\bar{f}^2 x) \leq \varepsilon \} \not= \emptyset \Rightarrow BP_\varepsilon (f) \not= \emptyset \text{ for any given } \varepsilon \in R_{0+}
\]

or, equivalently, we prove its equivalent contrapositive logic proposition, that is, \( BP_\varepsilon (f) = \emptyset \Rightarrow F^2 = \emptyset \) for any given \( \varepsilon \in R_{0+} \). Assume on the contrary that \( BP_\varepsilon (f) = \emptyset \Rightarrow F^2 = \emptyset \). Then, \( d(x, f x) \geq D + \varepsilon \); \( \forall x \in A \cup B \) and \( d(x, f^2 x) < \varepsilon_1 \) for some \( \varepsilon \in R_{0+} \) and some \( x \in A \cup B \) and any \( \varepsilon_1 \in R_{0+} \). Note that, although \( BP_\varepsilon (f) = \emptyset \) for \( \varepsilon \in R_{0+} \) is being assumed, \( BP_0 (f) \not= \emptyset \) so that there are \( z \in A \) and \( f z \in B \) such that \( d(z, f z) = D \) since \( f : A \cup B \to A \cup B \) has 0-best proximity points in \( A \) and in \( B \). As a result, one has for some \( x \in A \cup B \):

\[
d(f^2 x, f x) + \varepsilon_1 > d(x, f^2 x) + d(f^2 x, f x) \geq d(x, f x) \geq D + \varepsilon
\]

By applying the above chain of inequalities to the 0-best proximity points \( z \) and \( f z \) satisfying \( d(z, f z) = D \). It is now proved that \( d(f^2 z, f z) = D \). Assume not. Then, the following sequence of points \( z \to f z \to f^2 z \) is generated through \( f : A \cup B \to A \cup B \). If \( f^2 z = z \) then \( d(f^2 z, f z) = d(z, f z) = D \) holds. Assume that \( f^2 z \not= z \) with \( d(z, f^2 z) > \varepsilon_1 \); \( \forall \varepsilon_1 \in R_{0+} \). Then, it follows from (2.5),

\[
D < d(f z, f^2 z) \leq d(z, f z) + d(z, f^2 z) > \varepsilon_1 + D
\]

fails for \( \varepsilon_1 = 0 \) so that again \( d(f^2 z, f z) = d(z, f z) = D \). Thus, it follows that \( d(f^2 z, f z) = d(z, f z) = D \) so that \( D + \varepsilon_1 > D + \varepsilon \) from (2.2) and then \( \varepsilon_1 > \varepsilon \). But this constraint fails for the given \( \varepsilon \in R_{0+} \) and \( 0 \leq \varepsilon_1 \leq \varepsilon \) which contradicts that \( \varepsilon_1 \in R_{0+} \) is arbitrary.

**Remark 2.10.** The condition that \( A \) is approximatively compact with respect to \( B \) in Theorem 2.9 can be changed by \( B \) being approximatively compact with respect to \( A \) and the theorem remains valid. Note that if \( A \) and \( B \) are compact then each of them is approximatively compact with respect to each other. Since \( A \) and \( B \) are assumed closed then it suffices to assume then, in addition, bounded and to maintaining the result without invoking the approximative compactness assumption.

According to Remark 2.10, Theorem 2.9 leads to the subsequent result:

**Corollary 2.11.** Let \((X, d)\) be a metric space and let \( A \) and \( B \) be nonempty compact subsets of \( X \). Then \( f : A \cup B \to A \cup B \) is cyclic asymptotically regular if and only if \( f^2 : A \cup B \to A \cup B \) is asymptotically regular.
Theorem 2.9 and Corollary 2.11 are, respectively, equivalently to the following results:

**Theorem 2.12.** Assume that $A$ and $B$ are nonempty closed subsets of $X$ where $(X,d)$ be a metric space. Assume also that $A$ is approximatively compact with respect to $B$. Then $f : A \cup B \to A \cup B$ has the approximate best proximity point property if and only if $f^2 : A \cup B \to A \cup B$ has the approximate fixed point property. □

**Corollary 2.13.** Let $(X,d)$ be a metric space and let $A$ and $B$ be nonempty compact subsets of $X$. Then $f : A \cup B \to A \cup B$ has the approximate best proximity point property if and only if $f^2 : A \cup B \to A \cup B$ has the approximate fixed point property. □

**Definitions 2.14.** Let $(X,d)$ be a metric space and let $f : A \cup B \to A \cup B$ be a 2-cyclic self-mapping on the union of two nonempty subsets $A$ and $B$ of $X$. Then,

1. $f : A \cup B \to A \cup B$ has the approximate best proximity point property if $BP_\varepsilon(f) \neq \emptyset$ for all $\varepsilon \in R_{0+}$.
2. $f : A \cup B \to A \cup B$ has the approximate best proximity point property in $A$ if $BP_{A\varepsilon}(f) \neq \emptyset$ for all $\varepsilon \in R_{0+}$.
3. $f : A \cup B \to A \cup B$ has the $\varepsilon_0$-partial best proximity point property if $BP_\varepsilon(f) \neq \emptyset$ for all real $\varepsilon \geq \varepsilon_0$ and a given $\varepsilon_0 \in R_{0+}$.
4. $f : A \cup B \to A \cup B$ has the $\varepsilon_0$-partial best proximity point property in $A$ if $BP_{A\varepsilon}(f) \neq \emptyset$ for all real $\varepsilon \geq \varepsilon_0$ and a given $\varepsilon_0 \in R_{0+}$. □

**Theorem 2.15.** Let $(X,d)$ be a metric space and let $A$ and $B$ be nonempty bounded closed subsets of $X$ with $d(A,B) = D$. Assume that the cyclic self-mapping $f : A \cup B \to A \cup B$ satisfies the contractive condition:

$$d(f^2x,fy) \leq Kd(fx,y) + (1-K)(D + \delta(x))$$

(2.7)

for $x \in A \cup B$ and some $K \in [0,1)$, where $\delta(x) = \varepsilon_{0A}$ if $x \in A$ and $\delta(x) = \varepsilon_{0B}$ if $x \in B$. Then, $f : A \cup B \to A \cup B$ is cyclic asymptotically $\varepsilon_{0A}$-regular in $A$ and cyclic asymptotically $\varepsilon_{0B}$-regular in $B$ has both the $\varepsilon_{0A}$-partial best proximity point property in $A$ and the $\varepsilon_{0B}(\neq \varepsilon_{0A})$-partial best proximity point property in $B$. Also, $f^2 : A \cup B \to A \cup B$ has not the approximate fixed point property and, equivalently, it is not cyclic asymptotically regular.

**Proof:** Since $f : A \cup B \to A \cup B$ satisfies (2.7), it has both the $\varepsilon_{0A}$-partial best proximity point property in $A$ and the $\varepsilon_{0B}(\neq \varepsilon_{0A})$-partial best proximity point property in $B$ then one has for any $x \in A$ since $BP_{A\varepsilon}(f) \neq \emptyset$ for any real $\varepsilon \geq \varepsilon_{0A}$ and $BP_{B\varepsilon}(f) \neq \emptyset$ for any real $\varepsilon \geq \varepsilon_{0B}$:

\[ \]
Approximations of best proximity points

\[ d\left(f^{2n}x, f^{2n+1}x\right) \to D + \epsilon_{0A} \quad ; \quad d\left(f^{2n+1}x, f^{2n+2}x\right) \to D + \epsilon_{0B} \quad \text{as} \quad n \to \infty \quad (2.8) \]

since \( f : A \cup B \to A \cup B \) is cyclic asymptotically \( \epsilon_{0A} \)-regular in \( A \) and cyclic asymptotically \( \epsilon_{0B} \)-regular in \( B \).

It follows by using the triangle inequality involving \( d\left(f^{2n}x, f^{2n+1}x\right), d\left(f^{2n}x, f^{2n+2}x\right) \)
\( d\left(f^{2n+1}x, f^{2n+2}x\right) \) and (2.8) that

\[
\liminf_{n \to \infty} d\left(f^{2n}x, f^{2n+2}x\right) \geq |\epsilon_{0A} - \epsilon_{0B}| > 0
\]

and then \( f^2 : A \cup B \to A \cup B \) neither has the approximate fixed point property nor it is cyclic asymptotically regular. \( \square \)

**Corollary 2.16.** Let \((X, d)\) be a metric space and let \( A \) and \( B \) be nonempty closed subsets of \( X \) with \( d(A, B) = D \). Assume that there are nonempty closed sets \( A' \subseteq A \) and \( B' \subseteq B \) and that the cyclic self-mapping \( f : A \cup B \to A \cup B \) satisfies the contractive condition:

\[
d\left(f^2x, fx\right) \leq Kd(fx, x) + (1 - K)(D + 2\delta)
\]

\[(2.9)\]

; \( \forall x \in A \cup B \) and some \( K \in [0, 1) \) with \( 0 \leq \delta \leq \min(diam A, diam B) \)

Then, \( f : A \cup B \to A \cup B \) has the \( \epsilon \)-partial approximate best proximity point property for any real \( \epsilon \in [\epsilon_0, 2\delta] \) and some real \( \epsilon_0 \in R_{0^+} \).

**Proof:** One has from (2.9) that

\[
D \leq d\left(f^{n+1}x, f^nx\right) \leq K^n d(fx, x) + (1 - K^n)(D + 2\delta)
\]

\[(2.10)\]

; \( \forall x \in A \cup B \). Thus,

\[
D \leq \liminf_{n \to \infty} d\left(f^{n+1}x, f^nx\right) \leq \limsup_{n \to \infty} d\left(f^{n+1}x, f^nx\right) \leq D + 2\delta
\]

\[(2.11)\]

; \( \forall x \in A \cup B \). Then, \( BP_{\epsilon}(f) \cap BP_{\epsilon}A(f) \cap BP_{\epsilon}B(f) \neq \emptyset \) for any real \( \epsilon \in [\epsilon_0, 2\delta] \) and some real \( \epsilon_0 \in R_{0^+} \). \( \square \)

**Remark 2.17.** Note that Corollary 2.16 does not guarantee that \( f : A \cup B \to A \cup B \) is cyclic asymptotically \( \epsilon \)-regular for \( \epsilon \in [\epsilon_0, 2\delta] \) and some \( \epsilon_0 \in R_{0^+} \); since one does not conclude from (2.11) that \( d\left(f^{n+1}x, f^nx\right) \) converges to a limit, unless \( \delta = 0 \), \( A' = A \) and \( B' = B \) even if \((X, d)\) is complete. \( \square \)

**Remark 2.18.** The decay of the distances between consecutive points of sequences under cyclic asymptotically regularity to its minimum value \( D \) is characterized in the sequel. Note that for \( \epsilon < 1 \), and since \( K < 1 \),
where
\[ n_0 = n_0(e, x) = \min \left\{ z \in \mathbb{Z}_{0+} : z \geq \frac{\ln(d(x, f(x)) - D) + |\ln \varepsilon|}{|\ln K|} \right\} \] (2.12)
since it follows from (2.2) for all \( n \geq n_0 \) that
\[ K^n (d(x, f(x)) - D) + D \leq D + \varepsilon \] (2.13)
In the same way, one has for any given \( h \in \mathbb{R}_+ \) that
\[ hD + \varepsilon \leq d \left( f^{n+1}x, f^n x \right) \leq (h+1)D + \varepsilon \] (2.14)
where
\[ n_0 = n_0(e, h, x) = \min \left\{ z \in \mathbb{Z}_{0+} : z \geq \frac{\ln(d(x, f(x)) - D) - \ln h - \ln D + |\ln \varepsilon|}{|\ln K|} \right\} \] (2.15)
\[ n_0 = n_0(e, h, x) = \max \left\{ z \in \mathbb{Z}_{0+} : z \leq \frac{\ln(d(x, f(x)) - D) - \ln (h-1) - \ln D + |\ln \varepsilon|}{|\ln K|} \right\} \] (2.16)
\[ hD + \varepsilon \leq K^n (d(x, f(x)) - D) + D \leq (h+1)D + \varepsilon \] (2.17)

**Example 2.19.** The observations of Remark 2.18 can be extended to the case that the contractive constant is varying and can either equalize or exceed unity at certain points. Then, (2.1) becomes modified as follows:
\[ d \left( f^{n+1}x, f^n x \right) \leq \sum_{i=1}^{n} d(f^{n-i}x, f^{n-i-1}x) + (1-K_n)D \]
(2.17)
\[ , \; \forall x \in A \cup B , \; \forall n \in \mathbb{Z}_{+} \] and, if \( g = \left( \prod_{j=1}^{n_0} K_j \right) < 1 \) and \( \left( \prod_{j=1}^{n_0} K_j \right) \leq g \) for some given \( \varepsilon \in \mathbb{R}_+ \), all \( n \geq n_0 \) if there is \( n_0 \in \mathbb{Z}_{0+} \) defined by
\[ n_0 = n_0(e, x) = \min \left\{ \zeta \in \mathbb{Z}_{0+} : \zeta \geq \frac{\ln(d(x, f(x)) - D) + |\ln \varepsilon|}{\sum_{i=1}^{n} \ln K_i} \right\} \] (2.18)
then
\[ d \left( f^{n+1}x, f^n x \right) \leq D + \varepsilon \] for all \( n \geq n_0 \) in \( \mathbb{Z}_{0+} \) and any \( x \in A \cup B \). This is obvious since there is a real constant \( K \in [0, 1) \) such that \( K^n \leq K^{n_0} = g \), equivalently, \( -n|\ln K| \leq -n_0|\ln K| = \ln g \).
The above discussion leads to the fact that \( f: A \cup B \to A \cup B \) has the \( \varepsilon_1 \)-approximate best proximity point property for any \( \varepsilon_1 \geq \varepsilon \) and it also has \( \varepsilon \)-approximate best proximity points in \( A \) and \( B \) (i.e. their mutual distance is non larger than \( D + \varepsilon \)) if (2.18) is well posed for all \( x \in A \cup B \). However, the cyclic asymptotic regularity and the approximate best proximity point properties are not guaranteed if \( \liminf_{n \to \infty} \left( \prod_{j=1}^{n} K_j \right) > 0 \) while such properties are guaranteed if the limit \( \lim_{n \to \infty} \left( \prod_{j=1}^{n} K_j \right) \) exists and is zero.

Roughly speaking, in this last case, there is no convergence of the sequences to the best proximity points and convergence of distances to \( D \).

The above example concludes the following result:

**Proposition 2.20.** Assume that \( \left\{ z \in Z_{0+} : z \geq \sup_{x \in A \cup B} \left( \sum_{i=1}^{n} \ln K_i \right) \right\} \neq \emptyset \). Then, the cyclic self-mapping \( f: A \cup B \to A \cup B \) has the \( \varepsilon_1 \)-partial approximate best proximity point property for any \( \varepsilon_1 \geq \varepsilon \) and it also has \( \varepsilon \)-approximate best proximity points in \( A \) and \( B \). Then, any sub-sequences \( \{ T^{2n} x \} \subset A \) and \( \{ T^{2n+1} x \} \subset B \) remain within closed balls around the best proximity points of \( A \) and \( B \) for all \( n \geq n_0 \) with

\[
 n_0 = n_0(\varepsilon, x) = \min \left\{ z \in Z_{0+} : z \geq \sup_{x \in A \cup B} \left( \frac{\ln(d(x, f x) - D) + \ln \varepsilon}{\sum_{i=1}^{n} \ln K_i} \right) \right\},
\]

\( \forall x \in A \). If, furthermore, \( \lim_{n \to \infty} \left( \prod_{j=1}^{n} K_j \right) = 0 \) then \( f: A \cup B \to A \cup B \) has the approximate best proximity point property if \( \text{BP}_{\varepsilon}(f) \neq \emptyset \) for all \( \varepsilon \in \mathbb{R}_{0+} \) and it is asymptotically regular. \( \Box \)

**Acknowledgements.** The author thanks the Basque Government and UPV/EHU by Grants GIC07143-IT-269-07 and UFI 2011/07.

**References**


Received: July 7, 2014