$\mu$-rgb-Continuous, $\mu$-rgb-Closed and $\mu$-rgb-Open Functions in a Generalized Topological Space

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Abstract
This study introduces the concept of $\mu$-regular generalized b-closed ($\mu$-rgb-closed) set in a generalized topological space $(X, \mu)$. Some properties and characterizations of $\mu$-rgb-continuous functions, $\mu$-rgb-closed functions, and $\mu$-rgb-open functions are also considered.

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1 Introduction

In 1937, Stone [8] introduced and investigated the regular open sets in a topological space. These sets are contained in the family of open sets since a set is regular open if it is equal to the interior of its closure. In 1982, Levine [7] introduced the concept of generalized closed set and discussed the

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properties of sets, closed and open maps, compactness, normal and separation axioms. Later in 1996, Andrijevic [1] gave a new type of generalized closed set in topological space called b-closed sets. The investigation on generalized of closed set has lead to significant contribution to the theory of separation axiom, generalization of continuity and covering properties.

In 2002, ´A Cs´ asz´ ar [4] introduced the concept of generalized topological space (or simply GT-space).

In this paper, a new class of closed set called the regular generalized b-closed set in a generalized topological space is introduced.

2 Preliminaries

Definition 2.1 [4] Let $X$ be a nonempty set. A collection $\mu$ of subsets of $X$ is a generalized topology (or briefly GT) on $X$ if it satisfies the following:

(O1) $\emptyset \in \mu$; and

(O2) If $\{M_i : i \in I\} \subseteq \mu$, then $\bigcup_{i \in I} M_i \in \mu$.

If $\mu$ is a GT on $X$, then $(X, \mu)$ is called a generalized topological space (or briefly GT-space), and the elements of $\mu$ are called $\mu$-open sets.

Definition 2.2 [4] Let $\mu$ be a GT on $X$. A subset $F$ of $X$ is said to be $\mu$-closed if the complement $F^c$ of $F$ is $\mu$-open.

Definition 2.3 [3] Let $(X, \mu)$ be a GT-space and let $A \subseteq X$. The $\mu$-closure of $A$, denoted by $c_\mu(A)$, is the intersection of all $\mu$-closed sets containing $A$.

Definition 2.4 [3] Let $(X, \mu)$ be a GT-space. A subset $A$ of $X$ is said to be $\mu$-b-open if $A \subseteq c_\mu(i_\mu(A)) \cup i_\mu(c_\mu(A))$.

The complement of a $\mu$-b-open set with respect to the set $X$ is said to be a $\mu$-b-closed) set in $X$.

Definition 2.5 [1] Let $(X, \mu)$ be a GT-space and $A \subseteq X$. The $\mu$-b-closure of $A$, denoted by $bc_\mu(A)$, is the intersection of all $\mu$-b-closed sets containing $A$.

Definition 2.6 A set $A$ is said to be:

(i.) $\mu$-regular open if $A = i_\mu(c_\mu(A))$;

(ii.) $\mu$-regular closed if $A = c_\mu(i_\mu(A))$;

(iii.) $\mu$-regular generalized closed (briefly $\mu$-rg closed) if $c_\mu(A) \subseteq U$ whenever $A \subseteq U$, where $U$ is $\mu$-regular open.
(v.) \( \mu \)-regular generalized open (briefly \( \mu \)-rg open) if the complement of \( A \) is \( \mu \)-rg-closed.

**Definition 2.7** A subset \( A \) of a GT-space \((X, \mu)\) is called a \( \mu \)-regular generalized b-closed set (briefly \( \mu \)-rgb-closed set) if \( bc_\mu(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \mu \)-regular open in \( X \).

### 2.1 \( \mu \)-Regular Generalized b-Continuous Functions

This section defines and gives some properties of \( \mu \)-rgb-continuous function.

**Definition 2.8** A function \( f : X \rightarrow Y \) is said to be \( \mu \)-rgb-continuous if for every \( \mu \)-open subset \( U \) of \( Y \), \( f^{-1}(U) \) is \( \mu \)-rgb-open in \( X \).

**Theorem 2.9** Every \( \mu \)-continuous function is \( \mu \)-rgb-continuous.

**Remark 2.10** The converse of Theorem 2.10 is not true.

**Theorem 2.11** If \( f : X \rightarrow Y \) is \( \mu \)-rgb-continuous and \( g : Y \rightarrow Z \) is \( \mu \)-continuous, then \( g \circ f : X \rightarrow Z \) is \( \mu \)-rgb-continuous.

**Theorem 2.12** \( f : X \rightarrow Y \) is \( \mu \)-rgb-continuous if and only if \( f^{-1}(F) \) is \( \mu \)-rgb-closed set in \( X \) for each \( \mu \)-closed set \( F \) in \( Y \).

**Proof:** (\( \Rightarrow \)) Let \( F \) be any \( \mu \)-closed set in \( Y \). Then \( Y \setminus F \) is \( \mu \)-open in \( Y \). Since \( f \) is \( \mu \)-rgb-continuous, \( f^{-1}(Y \setminus F) \) is \( \mu \)-rgb-open in \( X \). Hence, \( f^{-1}(F) \) is \( \mu \)-rgb-closed in \( X \).

(\( \Leftarrow \)) Let \( U \) be any \( \mu \)-open set in \( Y \). Then \( Y \setminus U \) is \( \mu \)-closed in \( Y \). By assumption, \( f^{-1}(Y \setminus U) \) is \( \mu \)-rgb-closed in \( X \). Hence, \( f^{-1}(U) \) is \( \mu \)-rgb-open. Therefore, \( f \) is \( \mu \)-rgb-continuous. \( \square \)

**Theorem 2.13** If \( f : X \rightarrow Y \) is \( \mu \)-rgb-continuous and \( A \subseteq X \), then
\[
f(\mu\text{-rgb-}c_\mu(A)) \subseteq c_\mu(f(A)).
\]

**Theorem 2.14** If \( f : X \rightarrow Y \) is \( \mu \)-rgb-continuous and \( B \subseteq Y \), then
\[
\mu\text{-rgb-}c_\mu(f^{-1}(B)) \subseteq f^{-1}(c_\mu(B)).
\]

**Theorem 2.15** If \( f : X \rightarrow Y \) is an \( \mu \)-rgb-continuous function, then for each \( x \in X \) and every \( \mu \)-open set \( W \) in \( Y \) containing \( f(x) \), there exists an \( \mu \)-rgb-open set \( O \) in \( X \) such that \( x \in O \) and \( f(O) \subseteq W \).

**Proof:** Let \( x \in X \) and let \( W \) be any \( \mu \)-open set in \( Y \) containing \( f(x) \). Thus, \( x \in f^{-1}(W) \). Let \( O = f^{-1}(W) \). Since \( f \) is \( \mu \)-rgb-continuous, \( O \) is \( \mu \)-rgb-open in \( X \) with \( x \in O \) and \( f(O) = f(f^{-1}(W)) \subseteq W \). \( \square \)
2.2 $\mu$-Regular Generalized b-Open and $\mu$-Regular Generalized b-Closed Functions

Definition 2.16 A function $f : X \to Y$ is said to be $\mu$-rgb-open if the image $f(A)$ is $\mu$-rgb-open in $Y$ for each $\mu$-open set $A$ in $X$.

Definition 2.17 A function $f : X \to Y$ is said to be $\mu$-rgb-closed if the image $f(A)$ is $\mu$-rgb-closed in $Y$ for each $\mu$-closed set $A$ in $X$.

Theorem 2.18 Every $\mu$-open ($\mu$-closed) function is $\mu$-rgb-open ($\mu$-rgb-closed).

Remark 2.19 The converse of Theorem 2.18 is not true.

Theorem 2.20 If $f : X \to Y$ is $\mu$-rgb-open and $A \subseteq X$, then
$$f(i_\mu(A)) \subseteq \mu$-rgb-i_\mu(f(A)).$$

Theorem 2.21 $f : X \to Y$ is $\mu$-rgb-open if and only if for every subset $S$ of $Y$ and for every $\mu$-closed set $F$ of $X$ containing $f^{-1}(S)$, there exists a $\mu$-rgb-closed set $K$ of $Y$ containing $S$ such that $f^{-1}(K) \subseteq F$.

Proof: ($\Rightarrow$) Let $S \subseteq Y$ and $F$ be a $\mu$-closed set of $X$ such that $f^{-1}(S) \subseteq F$. Now, $X\setminus F$ is a $\mu$-open set in $X$. It follows that $f(X\setminus F)$ is $\mu$-rgb-open in $Y$. Then $K = Y\setminus f(X\setminus F)$ is a $\mu$-rgb-closed set in $Y$. Hence, $f^{-1}(K) \subseteq F$.

($\Leftarrow$) By assumption, there exists a $\mu$-rgb-closed set $K$ of $Y$ such that $X\setminus f(U) \subseteq K$ and $f^{-1}(K) \subseteq X\setminus U$ so that $U \subseteq X\setminus f^{-1}(K)$. Since $Y\setminus K$ is $\mu$-rgb-open, $f(U)$ is $\mu$-rgb-open in $Y$. Therefore, $f$ is $\mu$-rgb-open.

Theorem 2.22 If $f : X \to Y$ is $\mu$-rgb-open and $B \subseteq Y$, then
$$f^{-1}(\mu$-rgb-$c_\mu(B)) \subseteq c_\mu(f^{-1}(B)).$$

Theorem 2.23 Let $f : X \to Y$ be bijective. Then $f$ is $\mu$-rgb-open if and only if $f$ is $\mu$-rgb-closed.

Proof: Let $f$ be a $\mu$-rgb-open function and let $D$ be a $\mu$-closed set in $X$. Then $X\setminus D$ is $\mu$-open in $X$ and $f(X\setminus D)$ is $\mu$-rgb-open in $Y$. Since $f$ is bijective, $Y\setminus f(D) = f(X\setminus D)$ is $\mu$-rgb-open in $Y$. Thus, $f(D)$ is $\mu$-rgb-closed in $Y$.

Conversely, let $f$ be a $\mu$-rgb-closed function and suppose that $O$ is a $\mu$-open set in $X$. Then $X\setminus O$ is $\mu$-closed and $Y\setminus f(O) = f(X\setminus O)$ is $\mu$-rgb-closed. Therefore, $f(O)$ is $\mu$-rgb-open in $X$.

Theorem 2.24 If a function $f : X \to Y$ is $\mu$-rgb-closed and $A \subseteq X$, then
$$\mu$-rgb-$c_\mu(f(A)) \subseteq f(c_\mu(A)).$$
On rgb-countinuous, rgb-closed and rgb-open functions

Proof: Since $A \subseteq c_\mu(A)$, $f(A) \subseteq f(c_\mu(A))$. Moreover, since $c_\mu(A)$ is $\mu$-closed in $X$, $f(c_\mu(A))$ is $\mu$-rgb-closed. Thus,

$$\mu\text{-rgb-}c_\mu(f(A)) \subseteq f(c_\mu(A)).$$

**Theorem 2.25** If $f : X \rightarrow Y$ is $\mu$-rgb-closed and $A$ is $\mu$-closed in $X$, then $f|_A : A \rightarrow Y$ is $\mu$-rgb-closed.

**Theorem 2.26** Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be mappings such that their composition $g \circ f : X \rightarrow Z$ is $\mu$-rgb-closed. If $f$ is continuous and surjective, then $g$ is $\mu$-rgb-closed.

**Theorem 2.27** If $f : X \rightarrow Y$ is a $\mu$-closed map and $g : Y \rightarrow Z$ is $\mu$-rgb-closed, then the composition $g \circ f : X \rightarrow Z$ is $\mu$-rgb-closed.

**Theorem 2.28** $f : X \rightarrow Y$ is $\mu$-rgb-closed if and only if for each subset $S$ of $Y$ and each $\mu$-open set $U$ containing $f^{-1}(S)$, there exists an $\mu$-rgb-open set $V$ of $Y$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof: Suppose that $f$ is $\mu$-rgb-closed. Let $S \subseteq Y$ and $U$ be a $\mu$-open set of $X$ such that $f^{-1}(S) \subseteq U$. Now, $X \setminus U$ is a $\mu$-closed set in $X$. Then $V = Y \setminus f(X \setminus U)$ is a $\mu$-rgb-open set in $Y$. Thus, $f(X \setminus U) \subseteq f(f^{-1}(Y \setminus S)) \subseteq Y \setminus S$. Hence $Y \setminus (Y \setminus S) \subseteq Y \setminus f(X \setminus U)$ implying that $S \subseteq V$ and $f^{-1}(V) = X \setminus f^{-1}(f(X \setminus U)) \subseteq X \setminus (X \setminus U) = U$. That is, $f^{-1}(V) \subseteq U$.

For the converse, suppose that $F$ is a $\mu$-closed set of $X$. It follows that $Y \setminus V \subseteq f(F) \subseteq f(X \setminus f^{-1}(V)) = Y \setminus f(f^{-1}(V)) \subseteq Y \setminus V$, which implies that $f(F) = Y \setminus V$. Since $Y \setminus V$ is $\mu$-rgb-closed, $f(F)$ is $\mu$-rgb-closed in $Y$. Therefore, $f$ is $\mu$-rgb-closed. \qed

**Theorem 2.29** For a bijection map $f : X \rightarrow Y$, the following statements are equivalent:

(a.) $f^{-1} : Y \rightarrow X$ is $\mu$-rgb-continuous.

(b.) $f$ is $\mu$-rgb-open.

(c.) $f$ is $\mu$-rgb-closed.

Proof: (a) $\iff$ (b). Let $U$ be any $\mu$-open set in $X$. By hypothesis,

$$(f^{-1})^{-1}(U) = f(U)$$

is $\mu$-rgb-open in $Y$. 

Conversely, let $A$ be any $\mu$-open set in $X$. Since $f$ is $\mu$-rgb-open, $f(A) = (f^{-1})^{-1}(A)$ is $\mu$-rgb-open in $Y$. Hence, $f^{-1}$ is $\mu$-rgb-continuous.

(a) $\Leftrightarrow$ (c). Let $B$ be any $\mu$-closed set in $X$. Then $X\setminus B$ is $\mu$-open in $X$. Since $f^{-1}$ is $\mu$-rgb-continuous, $(f^{-1})^{-1}(X\setminus B) = (f(X\setminus B))$ is $\mu$-rgb-open in $Y$. Since $f$ is bijective, $f(X\setminus B) = Y\setminus f(B)$ is $\mu$-rgb-open in $Y$. Hence, $f(B)$ is $\mu$-rgb-closed in $Y$.

Conversely, suppose that $D$ is any $\mu$-open set in $X$. Then, $X\setminus D$ is $\mu$-closed in $X$. By hypothesis, $f(X\setminus D) = Y\setminus f(D)$ is $\mu$-rgb-closed in $Y$. Thus, $(f^{-1})^{-1}(D) = f(D)$ is $\mu$-rgb-open in $Y$. Therefore, $f^{-1}$ is $\mu$-rgb-continuous.

(b) $\Leftrightarrow$ (c). Let $F$ be any $\mu$-closed set in $X$. Then $X\setminus F$ is $\mu$-open in $X$. By assumption, $f(X\setminus F)$ is $\mu$-rgb-open in $Y$. Since $f$ is bijective, $f(X\setminus F) = Y\setminus f(F)$ is $\mu$-rgb-open in $Y$. Hence, $f(F)$ is $\mu$-rgb-closed in $Y$.

Conversely, let $G$ be any $\mu$-open set in $X$. Then $X\setminus G$ is $\mu$-closed in $X$. By hypothesis, $f(X\setminus G) = Y\setminus f(G)$ is $\mu$-rgb-closed since $f$ is bijective. Hence, $f(G)$ is $\mu$-rgb-open in $Y$. \hfill $\square$

References


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