Hermite-Hadamard-like Type Inequalities for Differentiable Harmonically Quasi-convex Functions

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Abstract

In this paper, by setting up a generalized integral identity for differentiable functions, the author obtain some new upper bounds of Hermite-Hadamard type inequalities for differentiable harmonically convex functions.

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1 Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard’s inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \rightarrow R$ be a convex function and $a, b \in I$ with $a < b$. Then following double inequalities hold:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$
Hermite-Hadamard’s inequalities for convex functions have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [1, 2, 3, 8, 9, 18] and references therein.

Let us recall some definitions of several kinds of convex functions:

**Definition 1.** Let $I$ be an interval in $R$. Then $f: I \to R$ is said to be convex on $I$ if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

**Definition 2.** Let $I$ be an interval in $R_+ = (0, \infty)$. A function $f: I \to R$ is said to be harmonically convex on $I$ if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

holds, for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2) is reversed, then $f$ is said to be harmonically concave.

In [4], İmdat İşcan established the following result of the Hermite-Hadamard type for harmonically convex functions:

**Theorem 1.1.** Let $f: I \subseteq R_+ = (0, \infty) \to R$ be a harmonically convex function on an interval $I$ and $f \in L[a,b]$, where $a, b \in I$ with $a < b$.

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2} \leq \frac{f(a) + f(b)}{2}.$$  \hspace{1cm} (3)

Also, in [4], İmdat İşcan established some new Hermite-Hadamard type inequalities, which estimate the difference between the middle and the rightmost terms in (3), for harmonically convex functions:

**Theorem 1.2.** Let $f: I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior $I^0$ of an interval $I$ in $R_+ = (0, \infty)$ and $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically convex function on $[a,b]$ for $q \geq 1$, then the following inequality holds:

$$\left|\frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2}\right| \leq \frac{ab(b-a)}{2} \lambda_1 \left[\lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q\right]^\frac{1}{q},$$
where
\[
\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right),
\]
\[
\lambda_2 = -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right),
\]
\[
\lambda_3 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right)
= \lambda_1 - \lambda_2.
\]

Definition 3. A function \( f : I \subseteq (0, \infty) \to [0, \infty) \) is said to be harmonically quasi-convex, if the inequality
\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq \sup \{ f(x), f(y) \}
\]
for all \( x, y \in I \) and \( t \in [0,1] \).

In [6], İmdat Işcan et al. established the following theorem, which gives an upper bound for the approximation of the integral average \( \frac{ab}{b-a} \int_a^b f(u) \, du \) by the value \( f(x) \) at a point \( x \in [a,b] \):

Theorem 1.3. Let \( f : I \subseteq R_+ = (0, \infty) \to R \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) in \( R_+ = (0, \infty) \) and \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is harmonically quasi-convex on \([a,b]\) for \( q \geq 1 \), then for all \( x \in [a,b] \) the following inequality
\[
\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right|
\leq \frac{ab}{b-a} \left\{ (x-a)^2 \left( C_1(a, x, q, q) \sup \{ |f'(x)|^q, |f'(a)|^q \} \right)^{\frac{1}{q}}
+ (b-x)^2 \left( C_2(b, x, q, q) \sup \{ |f'(x)|^q, |f'(b)|^q \} \right)^{\frac{1}{q}} \right\},
\]
holds, where
\[
C_1(a, x, \vartheta, \rho) = \frac{\beta(\rho+1,1)}{x^{2\vartheta}} \frac{\mathcal{B}(2\vartheta, \rho+1; \rho+2; 1 - \frac{a}{x})}{\beta(\rho+1,1)}
\]
\[
C_2(b, x, \vartheta, \rho) = \frac{\beta(1, \rho+1)}{b^{2\vartheta}} \frac{\mathcal{B}(2\vartheta, 1; \rho+2; 1 - \frac{x}{b})}{\beta(1, \rho+1)}.
\]
\( \beta \) is Euler Beta function defined by
\[
\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \quad x, y > 0,
\]
and \( _2F_1 \) is hypergeometric function defined by the power series

\[
_2F_1[a, b, c, x] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n
\]

for \( |x| < 1 \). Here \((q)_n\) is the Pochhammer symbol, which is defined by

\[
(q)_n = \begin{cases} 
1, & n = 0 \\
q(q+1)\cdots(q+n-1), & n > 0.
\end{cases}
\]

In this paper, we give some generalized inequalities connected with the left and right parts of (3), as a result of this, we obtain some generalized Hermite-Hadamard-like type inequalities for differentiable harmonically quasi-convex functions by setting up an integral identity for differentiable functions.

## 2 Main results

In order to find some new inequalities of Hermite-Hadamard-like type inequalities connected with the left and right parts of (3) for functions whose derivatives are harmonically quasi-convex, we need the following lemma [16]:

**Lemma 1.** Let \( f : I \subseteq \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( a, b \in I \) with \( a < b \). If \( f' \in L([a, b]) \), then for \( n \geq 2 \) the following identity

\[
I(f)(a, b; n)
\]

holds, for \( t \in [0, 1] \), where \( A_t(a, b) = ta + (1 - t)b \).
Now we turn our attention to establish inequalities of Hermit-Hadamard-like type for differentiable harmonically convex functions.

**Theorem 2.1.** Let \( f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R} \) be a differentiable function on the interior \( I^0 \) of an interval \( I \) and \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'| \) is harmonically quasi-convex on \([a, b] \), then, for any \( n \geq 2 \) the following inequality

\[
|I(f)(a, b; n)| \leq \frac{ab(b-a)}{n^2} \left[ \left\{ \mu_{11}(a, b) \sup \{|f'(A_{\frac{1}{n}}(b, a)), |f'(a)|\} \right. \right.
\left. + \mu_{11}(b, a) \sup \{|f'(b)|, |f'(A_{\frac{1}{n}}(a, b))|\} \right]
\]

\[
+ \frac{(n-2)^2}{4} \left\{ \mu_{12}(a, b) \sup \{|f'(A_{\frac{1}{2}}(a, b))|, |f'(A_{\frac{1}{n}}(a, b))|\} \right.
\left. + \mu_{12}(b, a) \sup \{|f'(A_{\frac{1}{2}}(a, b))|, |f'(A_{\frac{1}{2}}(b, a))|\} \right] \]

holds, where

\[
\mu_{11}(a, b) = \frac{n}{a(b-a)} + \frac{n^2}{(b-a)^2} \ln \left[ \frac{A_{\frac{1}{n}}(b, a)}{A_{\frac{1}{n}}(b, a)} \right];
\]

\[
\mu_{12}(a, b) = \frac{4n}{(n-2)(b-a)(a+b)}
\]

\[
+ \frac{4n^2}{(n-2)^2(b-a)^2} \ln \left[ \frac{A_{\frac{1}{n}}(a, b)}{A_{\frac{1}{n}}(a, b)} \right].
\]

**Proof** From Lemma 1, we have

\[
I(f)(a, b; n)
\leq \frac{ab(b-a)}{n^2} \left[ \left\{ \int_0^1 \frac{t}{A_t^2(a, A_{\frac{1}{n}}(b, a))} \left| f' \left( \frac{A_{\frac{1}{n}}(a, b)}{A_t(a, A_{\frac{1}{n}}(b, a))} \right) \right| dt \right. \right.
\left. + \int_0^1 \frac{1-t}{A_t^2(A_{\frac{1}{n}}(a, b), b)} \left| f' \left( \frac{bA_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(a, b), b)} \right) \right| dt \right]
\]

\[
+ \frac{(n-2)^2}{4} \left\{ \int_0^1 \frac{t}{A_t^2(A_{\frac{1}{2}}(a, b), A_{\frac{1}{n}}(a, b))} \left| f' \left( \frac{A_{\frac{1}{2}}(a, b)A_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{2}}(a, b), A_{\frac{1}{n}}(a, b))} \right) \right| dt \right.
\left. + \int_0^1 \frac{1-t}{A_t^2(A_{\frac{1}{n}}(b, a), A_{\frac{1}{2}}(a, b))} \left| f' \left( \frac{A_{\frac{1}{n}}(b, a)A_{\frac{1}{2}}(a, b)}{A_t(A_{\frac{1}{n}}(b, a), A_{\frac{1}{2}}(a, b))} \right) \right| dt \right\} \]
Since \(|f'|\) is harmonically quasi-convex on \([a, b]\), we have

\[
I(f)(a, b; n) 
\leq \frac{ab(b - a)}{n^2} \left\{ \int_0^1 \frac{t}{A_n^2(a, A_n^2(b, a))} \, dt \sup \left\{ \left| f'(A_n^2(a, b)) \right|, \left| f'(a) \right| \right\} \right.
\]

\[
+ \int_0^1 \frac{1 - t}{A_n^2(a, A_n^2(b, a))} \, dt \sup \left\{ \left| f'(b) \right|, \left| f'(A_n^2(a, b)) \right| \right\}
\]

\[
+ \left( \frac{(n - 2)^2}{4} \right) \left\{ \mu_{11}(a, b) \sup \left\{ \left| f'(A_n^2(a, b)) \right|, \left| f'(a) \right| \right\} \right.
\]

\[
+ \mu_{11}(b, a) \sup \left\{ \left| f'(b) \right|, \left| f'(A_n^2(a, b)) \right| \right\}
\]

\[
+ \frac{(n - 2)^2}{4} \left\{ \mu_{12}(a, b) \sup \left\{ \left| f'(A_n^2(a, b)) \right|, \left| f'(A_n^2(b, a)) \right| \right\} \right.
\]

\[
+ \mu_{12}(b, a) \sup \left\{ \left| f'(A_n^2(a, b)) \right|, \left| f'(A_n^2(b, a)) \right| \right\}
\}

which completes the proof.

Theorem 2.2. Let \(f : I \subseteq R_+ = (0, \infty) \to R\) be a differentiable function on the interior \(I^0\) of an interval \(I\) and \(f' \in L([a, b])\), where \(a, b \in I\) with \(a < b\). If \(|f'|^q\) is harmonically quasi-convex on \([a, b]\) for \(q > 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\), then, for any \(n \geq 2\) the following inequality

\[
I(f)(a, b; n) 
\leq \frac{ab(b - a)}{n^2} \left\{ \left( \mu_{21} \left( \frac{1}{2} \right) \left( \sup \left\{ \left| f'(A_n^2(a, b)) \right|^q, \left| f'(a) \right|^q \right\} \right) \right)^\frac{1}{q}
\]

\[
+ \mu_{21}(a, b) \left( \sup \left\{ \left| f'(A_n^2(a, b)) \right|^q, \left| f'(b) \right|^q \right\} \right)^\frac{1}{q}
\]

\[
+ \left( \frac{(n - 2)^2}{4} \right) \left\{ \mu_{22} \left( \frac{1}{2} \right) \left( \sup \left\{ \left| f'(A_n^2(a, b)) \right|^q, \left| f'(A_n^2(b, a)) \right|^q \right\} \right) \right)^\frac{1}{q}
\]

\[
+ \mu_{22}(b, a) \left( \sup \left\{ \left| f'(A_n^2(a, b)) \right|^q, \left| f'(A_n^2(b, a)) \right|^q \right\} \right) \right)^\frac{1}{q}
\}

(4)
holds, where

\[
\begin{align*}
\mu_{21}(a, b) &= \frac{A_{n}^{-2p}(a, b)}{1 + p} \cdot 2F_1[2p, p + 1, p + 2, -\frac{b - a}{a + (n - 1)b}], \\
\mu_{22}(a, b) &= \frac{A_{n}^{-2p}(a, b)}{1 + p} \cdot 2F_1[2p, p + 1, p + 2, \frac{(n - 2)(b - a)}{2(a + (n - 1)b)}].
\end{align*}
\]

**Proof** From Lemma 1 and by the Hölder integral inequality, we have

\[
I(f)(a, b; n) \\
\equiv \left| \frac{1}{a + b} \left( bf(A_{n}^{1}(b, a)) + af(A_{n}^{1}(a, b)) \right) - \frac{ab}{b - a} \int_{a}^{b} \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{ab(b - a)}{n^2} \left\{ \left( \mu_{21}^{\frac{1}{2}}(b, a) \left( \int_{0}^{1} \left| f'(\frac{aA_{n}^{1}(b, a)}{A_{n}^{1}(a, (b, a))}) \right|^q \, dt \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \\
+ \mu_{21}^{\frac{1}{2}}(a, b) \left( \int_{0}^{1} \left| f'(\frac{bA_{n}^{1}(a, b)}{A_{n}^{1}(a, (b, a))}) \right|^q \, dt \right)^{\frac{1}{2}} \right\} \\
+ \frac{(n - 2)^2}{4} \left\{ \mu_{22}^{\frac{1}{2}}(a, b) \left( \int_{0}^{1} \left| f'(\frac{A_{n}^{1}(a, b)A_{n}^{1}(a, b)}{A_{n}^{1}(a, (b, a))}) \right|^q \, dt \right)^{\frac{1}{q}} \\
+ \mu_{22}^{\frac{1}{2}}(b, a) \left( \int_{0}^{1} \left| f'(\frac{A_{n}^{1}(b, a)A_{n}^{1}(a, b)}{A_{n}^{1}(a, (b, a))}) \right|^q \, dt \right)^{\frac{1}{q}} \right\}.
\]
\]

(5)
Theorem 2.3. Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior $I^0$ of an interval $I$ and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for $q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$, then, for any $n \geq 2$ the following inequality

$$I(f)(a,b;n) \leq \frac{ab(b-a)}{n^2} \left( \frac{1}{p+1} \right)^{\frac{2}{p}} \left\{ \mu_{31}(a,b) \sup \left\{ \left| f'(A_{\frac{1}{n}}(b,a)) \right|^q, \left| f'(a) \right|^q \right\} \right\}^{\frac{1}{q}}$$

$$+ \left( \mu_{31}(b,a) \sup \left\{ \left| f'(b) \right|^q, \left| f'(A_{\frac{1}{n}}(a,b)) \right|^q \right\} \right)^{\frac{1}{q}}$$

$$+ \frac{(n-2)^2}{4} \left\{ \left( \mu_{32}(a,b) \sup \left\{ \left| f'(A_{\frac{1}{n}}(a,b)) \right|^q, \left| f'(A_{\frac{1}{2}}(a,b)) \right|^q \right\} \right)^{\frac{1}{q}} \right\}$$

$$+ \left( \mu_{32}(b,a) \sup \left\{ \left| f'(A_{\frac{1}{2}}(a,b)) \right|^q, \left| f'(A_{\frac{1}{n}}(a,b)) \right|^q \right\} \right)^{\frac{1}{q}} \right\}. \tag{10}$$

holds, where

$$\mu_{31}(a,b) = \frac{n\{a^{1-2q} - A_{\frac{1}{n}}(b,a)\}}{(2q-1)(b-a)}$$

$$\mu_{32}(a,b) = \frac{2n\{2^{2q-1}(a+b)^{1-2q} - A_{\frac{1}{n}}^{1-2q}(a,b)\}}{(n-2)(2q-1)(b-a)}.$$
Hermite-Hadamard-like type inequalities

Proof From Lemma 1 and by the Hölder integral inequality, we have

\[ I(f)(a, b; n) \leq \frac{ab(b - a)}{n^2} \left[ \left( \int_0^1 t^p dt \right)^{\frac{1}{p}} \times \left( \int_0^1 \frac{1}{A_t^{2q}(a, A_{\frac{1}{n}}(b, a))} \left| f'\left( \frac{aA_{\frac{1}{n}}(b, a)}{A_t(a, A_{\frac{1}{n}}(b, a))} \right) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} + \left( \int_0^1 (1 - t)^p dt \right)^{\frac{1}{p}} \times \left( \int_0^1 \frac{1}{A_t^{2q}(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \left| f'\left( \frac{bA_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \right) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \right] + \frac{(n - 2)^2}{4} \left( \int_0^1 t^p dt \right)^{\frac{1}{p}} \times \left( \int_0^1 \frac{1}{A_t^{2q}(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \left| f'\left( \frac{A_{\frac{1}{n}}(b, a)A_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(b, a), A_{\frac{1}{n}}(a, b))} \right) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \right] \]

\[
= \frac{ab(b - a)}{n^2} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \times \left\{ \left( \int_0^1 \frac{1}{A_t^{2q}(a, A_{\frac{1}{n}}(b, a))} \left| f'\left( \frac{aA_{\frac{1}{n}}(b, a)}{A_t(a, A_{\frac{1}{n}}(b, a))} \right) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} + \left( \int_0^1 (1 - t)^p dt \right)^{\frac{1}{p}} \times \left( \int_0^1 \frac{1}{A_t^{2q}(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \left| f'\left( \frac{bA_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \right) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\} \]

\[ + \frac{(n - 2)^2}{4} \times \left\{ \left( \int_0^1 \frac{1}{A_t^{2q}(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \left| f'\left( \frac{A_{\frac{1}{n}}(b, a)A_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(b, a), A_{\frac{1}{n}}(a, b))} \right) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\}. \quad (11) \]
Since \(|f'|^q\) is harmonically quasi-convex on \([a, b]\) for \(q > 1\), we have

\[
\begin{align*}
(a) & \quad \int_0^1 \frac{1}{A_t^{2q}(a, A_{\frac{1}{n}}(b, a))} |f'\left(\frac{a A_{\frac{1}{n}}(b, a)}{A_t(a, A_{\frac{1}{n}}(b, a))}\right)|^q \, dt \\
 & \quad \leq \mu_{31}(a, b) \sup \left\{ |f'(A_{\frac{1}{n}}(b, a))|^q, |f'(a)|^q \right\}, \\
(b) & \quad \int_0^1 \frac{1}{A_t^{2q}(A_{\frac{1}{n}}(a, b), b)} |f'\left(\frac{b A_{\frac{1}{n}}(a, b)}{A_t(a, A_{\frac{1}{n}}(a, b))}\right)|^q \, dt \\
 & \quad \leq \mu_{31}(b, a) \sup \left\{ |f'(b)|^q, |f'(A_{\frac{1}{n}}(a, b))|^q \right\}, \\
(c) & \quad \int_0^1 \frac{1}{A_t^{2q}(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(b, a))} |f'\left(\frac{A_{\frac{1}{n}}(a, b) A_{\frac{1}{2}}(a, a)}{A_t(A_{\frac{1}{n}}(a, b), A_{\frac{1}{2}}(a, a))}\right)|^q \, dt \\
 & \quad \leq \mu_{32}(a, b) \sup \left\{ |f'(A_{\frac{1}{n}}(a, b))|^q, |f'(A_{\frac{1}{2}}(a, a))|^q \right\}, \\
(d) & \quad \int_0^1 \frac{1}{A_t^{2q}(A_{\frac{1}{n}}(b, a), A_{\frac{1}{n}}(a, b))} |f'\left(\frac{A_{\frac{1}{n}}(a, b) A_{\frac{1}{2}}(a, a)}{A_t(A_{\frac{1}{n}}(b, a), A_{\frac{1}{2}}(a, a))}\right)|^q \, dt \\
 & \quad \leq \mu_{32}(b, a) \sup \left\{ |f'(A_{\frac{1}{2}}(a, b))|^q, |f'(A_{\frac{1}{n}}(a, b))|^q \right\}.
\end{align*}
\]

By substituting (12)–(15) in (11), we get the desired result (10).

**Theorem 2.4.** Let \(f : I \subseteq R_+ = (0, \infty) \rightarrow R\) be a differentiable function on the interior \(I^0\) of an interval \(I\) and \(f' \in L([a, b])\), where \(a, b \in I\) with \(a < b\). If \(|f'|^q\) is harmonically quasi-convex on \([a, b]\) for \(q \geq 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\), then, for any \(n \geq 2\) the following inequality

\[
I(f)(a, b; n) \leq \frac{ab(b - a)}{n^2} \left[ \left\{ \mu_{11}(a, b) \left( \sup \left\{ |f'(A_{\frac{1}{n}}(b, a))|^q, |f'(a)|^q \right\} \right)^{\frac{\frac{2}{p} + \frac{1}{q}}{\frac{1}{p}} + \frac{\frac{2}{q} + \frac{1}{p}}{\frac{1}{q}}} \\
+ \mu_{11}(b, a) \left( \sup \left\{ |f'(A_{\frac{1}{n}}(a, b))|^q, |f'(b)|^q \right\} \right)^{\frac{\frac{2}{p} + \frac{1}{q}}{\frac{1}{p}} + \frac{\frac{2}{q} + \frac{1}{p}}{\frac{1}{q}}} \right] + \frac{(n - 2)^2}{4} \left\{ \mu_{12}(a, b) \left( \sup \left\{ |f'(A_{\frac{1}{n}}(b, a))|^q, |f'(A_{\frac{1}{2}}(a, b))|^q \right\} \right)^{\frac{\frac{2}{q} + \frac{1}{p}}{\frac{1}{q}}} \\
+ \mu_{12}(b, a) \left( \sup \left\{ |f'(A_{\frac{1}{2}}(b, a))|^q, |f'(A_{\frac{1}{n}}(a, b))|^q \right\} \right)^{\frac{\frac{2}{p} + \frac{1}{q}}{\frac{1}{p}} + \frac{\frac{2}{q} + \frac{1}{p}}{\frac{1}{q}}} \right\} \right]\]

holds, where \(\mu_{2i}, i = 1, 2, 3, 4\) are defined in Theorem 2.1.
Proof From Lemma 1 and by the Hölder integral inequality, we have

\[ I(f)(a, b; n) \leq \frac{ab(b - a)}{n^2} \left\{ \left( \int_0^1 \frac{t}{A^2_n(a, A_{2n}^l(b, a))} \left| f'(\frac{aA_{2n}^l(b, a)}{A_1(a, A_{2n}^l(b, a))}) \right| \, dt \right)^{\frac{1}{p}} \right. \\
+ \left. \left( \int_0^1 \frac{1 - t}{A^2_n(A_{2n}^l(a, b), A_{2n}^l(a, b))} \left| f'(\frac{bA_{2n}^l(a, b)}{A_1(A_{2n}^l(a, b), A_{2n}^l(a, b))}) \right| \, dt \right)^{\frac{1}{p}} \right\} \\
+ \frac{(n - 2)^2}{4} \left\{ \left( \frac{t}{A^2_n(A_{2n}^l(a, b), A_{2n}^l(a, b))} \left| f'(\frac{A_{2n}^l(a, b)A_{2n}^l(b, a)}{A_1(A_{2n}^l(a, b), A_{2n}^l(a, b))}) \right| \, dt \right)^{\frac{1}{q}} \right. \]

\[ \leq \frac{ab(b - a)}{n^2} \left\{ \left( \int_0^1 \frac{t}{A^2_n(a, A_{2n}^l(b, a))} \left| f'(\frac{aA_{2n}^l(b, a)}{A_1(a, A_{2n}^l(b, a))}) \right| \, dt \right)^{\frac{1}{p}} \right. \\
+ \left. \left( \int_0^1 \frac{1 - t}{A^2_n(A_{2n}^l(a, b), A_{2n}^l(a, b))} \left| f'(\frac{bA_{2n}^l(a, b)}{A_1(A_{2n}^l(a, b), A_{2n}^l(a, b))}) \right| \, dt \right)^{\frac{1}{p}} \right\} \\
+ \frac{(n - 2)^2}{4} \left\{ \left( \frac{t}{A^2_n(A_{2n}^l(a, b), A_{2n}^l(a, b))} \left| f'(\frac{A_{2n}^l(a, b)A_{2n}^l(b, a)}{A_1(A_{2n}^l(a, b), A_{2n}^l(a, b))}) \right| \, dt \right)^{\frac{1}{q}} \right. \\
+ \left. \left( \frac{1 - t}{A^2_n(A_{2n}^l(a, b), A_{2n}^l(a, b))} \left| f'(\frac{A_{2n}^l(a, b)A_{2n}^l(b, a)}{A_1(A_{2n}^l(a, b), A_{2n}^l(a, b))}) \right| \, dt \right)^{\frac{1}{q}} \right\} \\
= \frac{ab(b - a)}{n^2} \left\{ \left( \frac{t}{A^2_n(a, A_{2n}^l(b, a))} \left| f'(\frac{aA_{2n}^l(b, a)}{A_1(a, A_{2n}^l(b, a))}) \right| \, dt \right)^{\frac{1}{p}} + \left( \frac{1 - t}{A^2_n(A_{2n}^l(a, b), A_{2n}^l(a, b))} \left| f'(\frac{bA_{2n}^l(a, b)}{A_1(A_{2n}^l(a, b), A_{2n}^l(a, b))}) \right| \, dt \right)^{\frac{1}{p}} \\
+ \frac{(n - 2)^2}{4} \left\{ \left( \frac{t}{A^2_n(A_{2n}^l(a, b), A_{2n}^l(a, b))} \left| f'(\frac{A_{2n}^l(a, b)A_{2n}^l(b, a)}{A_1(A_{2n}^l(a, b), A_{2n}^l(a, b))}) \right| \, dt \right)^{\frac{1}{q}} + \left( \frac{1 - t}{A^2_n(A_{2n}^l(a, b), A_{2n}^l(a, b))} \left| f'(\frac{A_{2n}^l(a, b)A_{2n}^l(b, a)}{A_1(A_{2n}^l(a, b), A_{2n}^l(a, b))}) \right| \, dt \right)^{\frac{1}{q}} \right\} \right\}.
\[
+ \mu_{11}^{\frac{1}{2}}(b, a) \left( \int_0^1 \frac{1 - t}{A_t^2(A_{\frac{1}{n}}(b, a), b)} \left| f' \left( \frac{bA_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(a, b)), b} \right) \right|^q dt \right)^{\frac{1}{q}}
+ \frac{(n - 2)^2}{4} \times \left\{ \mu_{12}^{\frac{1}{2}}(a, b) \left( \int_0^1 \frac{t}{A_t^2(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \left| f' \left( \frac{A_{\frac{1}{n}}(a, b)A_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \right) \right|^q dt \right)^{\frac{1}{q}}
+ \mu_{12}^{\frac{1}{2}}(b, a) \left( \int_0^1 \frac{1 - t}{A_t^2(A_{\frac{1}{n}}(b, a), A_{\frac{1}{n}}(b, a))} \left| f' \left( \frac{A_{\frac{1}{n}}(b, a)A_{\frac{1}{n}}(b, a)}{A_t(A_{\frac{1}{n}}(b, a), A_{\frac{1}{n}}(b, a))} \right) \right|^q dt \right)^{\frac{1}{q}} \right\}.
\]

Since \(|f'|^q \) is harmonically quasi-convex on \([a, b]\) for \(q > 1\), we have

\[(a) \int_0^1 \frac{t}{A_t^2(a, A_{\frac{1}{n}}(b, a))} \left| f' \left( \frac{aA_{\frac{1}{n}}(b, a)}{A_t(a, A_{\frac{1}{n}}(b, a))} \right) \right|^q dt \leq \mu_{11}(a, b) \sup \left\{ \left| f'(A_{\frac{1}{n}}(b, a)) \right|^q, \left| f'(a) \right|^q \right\}, \quad (18)\]

\[(b) \int_0^1 \frac{1 - t}{A_t^2(A_{\frac{1}{n}}(b, a), b)} \left| f' \left( \frac{bA_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(a, b)), b} \right) \right|^q dt \leq \mu_{11}(a, b) \sup \left\{ \left| f'(b) \right|^q, \left| f'(A_{\frac{1}{n}}(a, b)) \right|^q \right\}, \quad (19)\]

\[(c) \int_0^1 \frac{t}{A_t^2(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \left| f' \left( \frac{A_{\frac{1}{n}}(a, b)A_{\frac{1}{n}}(a, b)}{A_t(A_{\frac{1}{n}}(a, b), A_{\frac{1}{n}}(a, b))} \right) \right|^q dt \leq \mu_{12}(a, b) \sup \left\{ \left| f'(A_{\frac{1}{n}}(a, b)) \right|^q, \left| f'(A_{\frac{1}{n}}(b, b)) \right|^q \right\}, \quad (20)\]

\[(d) \int_0^1 \frac{1 - t}{A_t^2(A_{\frac{1}{n}}(b, a), A_{\frac{1}{n}}(b, a))} \left| f' \left( \frac{A_{\frac{1}{n}}(b, a)A_{\frac{1}{n}}(b, a)}{A_t(A_{\frac{1}{n}}(b, a), A_{\frac{1}{n}}(b, a))} \right) \right|^q dt \leq \mu_{12}(b, a) \sup \left\{ \left| f'(A_{\frac{1}{n}}(b, a)) \right|^q, \left| f'(A_{\frac{1}{n}}(b, a)) \right|^q \right\}. \quad (21)\]

By substituting (18)-(21) in (17), we get the desired result (16).

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