Secure Domination and Secure Total
Domination in the Composition $G[K_n]$

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Abstract

In this paper we consider the concepts of secure domination and
secure total domination in a graph. We rectify a bit an interesting result
obtained by Benecke et al. and then state some consequences which

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are due them. Also, we characterize the secure dominating and secure total dominating sets in the composition $G[K_n]$, where $G$ is a connected graph and $K_n$ is the complete graph of order $n$, and gave upper bounds of the secure domination and secure total domination numbers of this graph.

**Mathematics Subject Classification:** 05C69

**Keywords:** domination, total domination, secure domination, secure total domination, composition

1 Introduction

Recently, Benecke et al. in [1] and Cockayne et al. in [3] introduced new strategies for placing guards in order to protect a system or network. These strategies give rise to new variants of the standard domination concept. A number of interesting results are found in [1] and [3] concerning the concepts and parameters. Other variants of the domination can be found in [2] and [4].

Let $G = (V(G), E(G))$ be a connected graph and $v \in V(G)$. The neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. If $X \subseteq V(G)$, then the open neighborhood of $X$ is the set $N_G[X] = N[X] = \bigcup_{v \in X} N_G(v)$. The closed neighborhood of $X$ is $N_G[X] = N[X] = X \cup N(X)$.

A subset $X$ of $V(G)$ is a dominating set in $G$ if for every $v \notin (V(G) \setminus X)$, there exists $x \in X$ such that $xv \in E(G)$, i.e., $N[X] = V(G)$. It is a total dominating set (a tds) if $N(X) = V(G)$. A (total) dominating set $X$ is a secure (total) dominating set if for every $u \in V(G) \setminus X$, there exists $v \in X$ such that $uv \in E(G)$ and $[X \setminus \{v\}] \cup \{u\}$ is a (total) dominating set. In this case, we say that $v$ $X$-defends $u$ or $u$ is $X$-defended by $v$. The domination number $\gamma(G)$ (total domination number $\gamma_t(G)$ and secure (total) domination number $\gamma_{st}(G)$ ($\gamma_{st}(G)$) ) of $G$ is the smallest cardinality of a dominating (resp., total dominating and secure (total) dominating) set in $G$.

A vertex $u \in X$, where $X \subseteq V(G)$, is an $X$-internal private neighbor ($X$-ipn) of $v \in X$ if $N_G(u) \cap X = \{v\}$. A vertex $w \in V(G) \setminus X$ is an $X$-external private neighbor ($X$-epn) of $v \in X$ if $N_G(w) \cap X = \{v\}$. The set of all the $X$-ipns (respectively $X$-epns) of $v$ is denoted by $ipn(v, X)$ (respectively $epn(v, X)$).

In this paper we try to rectify a bit a result obtained by Benecke et al. in [1] and then state some quick consequences of the result by the authors. We also study secure (total) dominating sets in composition of $G$ and $K_n$, and gave an upper bound for the secure (total) domination number of this graph.
Secure Domination in the Composition $G[K_n]$

The composition $G[H]$ of two graphs $G$ and $H$ is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

Observe that any non-empty subset $C$ of $V(G) \times V(H)$ (in fact, any set of ordered-pairs) can be written as $C = \cup_{x \in S}\{x\} \times T_x$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Henceforth, we shall use this form to denote any subset $C$ of $V(G) \times V(H)$.

The following result characterizes secure dominating sets in the composition $G[K_n]$.

**Theorem 2.1** Let $G$ be a connected graph and $n \geq 2$. Then $C = \cup_{x \in S}\{x\} \times T_x$, where $S \subseteq V(G)$ and $T_x \subseteq V(K_n)$ for each $x \in S$, is a secure dominating set in $G[K_n]$ if and only if either

(i) $S$ is a secure dominating set in $G$ or

(ii) $S$ is a dominating set in $G$ satisfying the following property:

(a) $|T_x| \geq 2$ for every $x \in S$ such that $epn(x; S) \neq \emptyset$ and $\langle epn(x; S) \rangle$ is not complete; and

(b) there exists $w \in N_G(z) \cap S$ with $|T_w| \geq 2$ for every $z \in V(G) \setminus S$ such that $epn(y; S)$ is not contained in $N_G[z]$ for every $y \in N_G(z) \cap S$.

**Proof:** Suppose $C = \cup_{x \in S}\{x\} \times T_x$ is a secure dominating set in $G[K_n]$. Let $u \in V(G) \setminus S$ and pick $b \in V(K_n)$. Since $C$ is a dominating set in $G[K_n]$, there exists $(x, c) \in C$ such that $(x, c)(u, b) \in E(G[K_n])$. This implies that $x \in S$ and $u \in N_G(x)$. This shows that $S$ is a dominating set in $G$. If $S$ is a secure dominating set, then we are done. Suppose now that $S$ is not a secure dominating set in $G$. Let $x \in S$ such that $\langle epn(x; S) \rangle$ is not complete. Then there exist distinct vertices $y, z \in epn(x; S)$ such that $yz \notin E(G)$. Let $a \in V(K_n)$. Then $(y, a), (z, a) \notin C$ and $(y, a)(z, a) \notin E(G[K_n])$. Since $C$ is a secure dominating set of $G[K_n]$, there exists $(x, p) \in C$ such that $(y, a)(x, p) \in E(G[K_n])$ and $C_1 = [(y, a)] \cup \{(x, a)\}$ is dominating in $G[K_n]$. Thus, since $(y, a) \notin C_1$ and $z \in epn(x; S)$, there exists $q \in T_x \setminus \{p\}$ such that $(x, q)(z, a) \in E(G[K_n])$. This shows that $|T_x| \geq 2$. Next, let $z \in V(G) \setminus S$ such that $epn(y; S)$ is not contained in $N_G(z)$ for every $y \in N_G(z) \cap S$. Pick $t \in V(K_n)$. Since $C$ is a secure dominating set and $(z, t) \notin C$, there exists $(w, c) \in C$ such that $(z, t)(w, c) \in E(G[K_n])$ and $C^* = [(z, t)] \cup \{(z, t)\}$ is dominating set in $G[K_n]$. By assumption, $epn(w; S)$ is not contained in $N_G[z]$. Hence there exists $u \in epn(w; S) \setminus N_G[z]$. This implies that $u \neq z$ and $uz \notin E(G)$. Since $C^*$ is a dominating set in $G[K_n]$, $(u, t) \notin C_1$, and
Let $G$ be a connected graph and $n \geq 2$. Then

$$\gamma_s(G[K_n]) \leq \min\{2\gamma(G), \gamma_s(G)\}.$$ 

Proof: Suppose $S$ is a minimum dominating set in $G$. Let $a, b \in V(K_n)$, where $a \neq b$ and set $T_x = \{a, b\}$ for every $x \in S$. Then, by Theorem 3.1, $C = \cup_{x \in S} \{x\} \times T_x$ is a secure dominating set in $G[K_n]$. Thus, $\gamma_s(G[K_n]) \leq |C| = 2|S| = 2\gamma(G)$. If $S$ is a minimum secure dominating set in $G$, set $T_x = \{a\}$ for each $x \in S$. Then again, by Theorem 3.1, $C = \cup_{x \in S} \{x\} \times T_x$ is a secure dominating set in $G[K_n]$. Thus, $\gamma_s(G[K_n]) \leq |C| = |S| = \gamma_s(G)$. Therefore,

$$\gamma_s(G[K_n]) \leq \min\{2\gamma(G), \gamma_s(G)\}.$$ 

This proves the result. □
Example 2.3 \( \gamma_s(P_3[K_n]) = 2 = \gamma_s(P_3) = 2\gamma(P_3) \) and \( \gamma_s(P_4[K_n]) = \gamma_s(P_4) = 2 \neq 4 = 2\gamma(P_4) \) for each \( n \geq 2 \).

The first result is due to Benecke et al. [1].

Proposition 2.4 Let \( X \) be a total dominating set in graph \( G \). A vertex \( v \) \( X \)-defends \( u \) if and only if

(i) \( epn(v, X) = \emptyset \), and

(ii) \( ipn(v, X) \subseteq N_G(u) \).

The following simple example will show that (i) is not the correct condition:

![Graph Image]

Let \( X = \{x, y, v\} \). Clearly, \( X \) is a total dominating set in \( G \). Also, \( uv \in E(G) \) and \( [X\setminus\{v\}] \cup \{u\} = \{x, y, u\} \) is a total dominating set in \( G \). It follows that \( v \) \( X \)-defends \( u \). However, \( epn(v, X) = \{w\} \neq \emptyset \).

The following result gives the correct characterization.

Theorem 2.5 Let \( X \) be a total dominating set in a connected graph \( G \), \( v \in X \), and \( u \in V(G) \setminus X \). Then \( v \) \( X \)-defends \( u \) if and only if \( epn(v, X) \) and \( ipn(v, X) \) are contained in \( N_G(u) \).

Proof: Suppose \( v \) \( X \)-defends \( u \). Let \( Y = [X\setminus\{v\}] \cup \{u\} \). Suppose first that \( epn(v, X) \not\subseteq N_G(u) \), say \( w \in epn(v, X) \setminus N_G(u) \). If \( u = w \), then there exists no \( y \in Y \) such that \( yw \in E(G) \) since \( N_G(w) \cap X = \{v\} \). It follows that \( Y \) is not a total dominating set. If \( u \neq w \), then there exists no \( y \in Y \) such that \( yw \in E(G) \) since \( N_G(w) \cap X = \{v\} \) and \( w \notin N_G(u) \). This implies that \( Y \) is not a total dominating set. Thus, in either case, we obtain a contradiction. Therefore \( epn(v, X) \subseteq N_G(u) \).

Next, suppose that \( ipn(v, X) \not\subseteq N_G(u) \), say \( z \in ipn(v, X) \setminus N_G(u) \). Since \( N_G(z) \cap X = \{v\} \), there exists no \( a \in Y \) such that \( az \in E(G) \). This implies that \( Y \) is not a total dominating set, contrary to our assumption. Therefore \( ipn(v, X) \subseteq N_G(u) \).
For the converse, suppose $X$ is a tds in $G$ and $epn(v,X)$ and $ipn(v,X)$ are contained in $N_G(u)$. Let $z \in V(G) \setminus Y$, where $Y = [X \setminus \{v\} \cup \{u\}]$. If $z = v$, then $zu \in E(G)$. So suppose $z \neq v$. Then $z \notin X$ and $z \neq u$. Since $X$ is a dominating set, $N_G(z) \cap X \neq \emptyset$. If $z \in epn(v,X)$, then $uz \in E(G)$ by assumption. If $z \notin epn(v,X)$, then $N_G(z) \cap X \neq \{v\}$; hence there exists $y \in X \setminus \{v\} \subset Y$ such that $xy \in E(G)$. In any of the above cases, we find that $Y$ is a dominating set in $G$.

Finally, let $w \in Y$. If $w = u$, then there exists $b \in X \setminus \{v\} \subset Y$ such that $bw \in E(G)$ since $X$ is a dominating set and $u \notin epn(v,X)$. Suppose $w \neq u$. Then $w \in X$. If $w \in ipn(v,X)$, then $xw \in E(G)$ for some $x \in X \setminus \{v\} \subset Y$. This shows that $Y$ is a total dominating set in $G$. Therefore, $v$ $X$-defends $u$. $\square$

The following results of Benecke et al. will still be immediate.

**Corollary 2.6** If $u \in epn(v,X)$ for some $v \in X$, then $u$ is not $X$-defended.

**Corollary 2.7** Let $X$ be a total dominating set in $G$. Then $X$ is a secure total dominating set if and only if

(i) $epn(v,X) = \emptyset$ for all $v \in X$, and

(ii) For each $u \in V(G) \setminus X$, there exists $v \in X \cap N(u)$ such that $ipn(v,X) \subseteq N(u)$.

## 3 Secure Total Domination in the Composition $G[K_n]$

The following result characterizes secure total dominating sets in the composition $G[K_n]$.

**Theorem 3.1** Let $G$ be a connected graph and $n \geq 2$. Then $C = \cup_{x \in S}\{x\} \times T_x$, where $S \subseteq V(G)$ and $T_x \subseteq V(K_n)$ for each $x \in S$, is a secure total dominating set in $G[K_n]$ if and only if either

(i) $S$ is a secure total dominating set in $G$ or

(ii) $S$ is a dominating set in $G$ satisfying the following properties:

(a) $|T_x| \geq 2$ for every $x \in S$ with $x \notin N_G(y)$ for all $y \in S$ or there exists $z \in N_G(x) \setminus S$ such that $z \notin N_G(w)$ for all $w \in (S \setminus \{x\})$; and

(b) for every $u \in V(G) \setminus S$ with $|N_G(u) \cap S| \geq 2$, there exists $z \in N_G(u) \cap S$ such that $|T_z| \geq 2$ or $|T_z| = 1$ and for every $q \in N_G(z) \cap S$, $uq \in E(G)$ or $|T_q| \geq 2$ or $q \in N_G(y)$ for some $y \in S \setminus \{z\}$.
Proof: Suppose $C = \bigcup_{x \in S}(\{x\} \times T_x)$ is a secure total dominating set in $G[K_n]$. Let $u \in V(G)\setminus S$ and pick $b \in V(K_n)$. Since $C$ is a dominating set in $G[K_n]$, then there exists $(x, c) \in C$ such that $(x, c)(u, b) \in E(G[K_n])$. This implies that $x \in S$ and $u \in N_{G}(x)$. This shows that $S$ is a dominating set in $G$. If $S$ is a secure total dominating set, then we are done. Suppose now that $S$ is not a secure total dominating set in $G$. Let $x \in S$ and consider the following cases:  

Case 1. Suppose $x \notin N_{G}(y)$ for all $y \in S$.  

Let $a \in T_x$. Since $(x, a) \in C$ and $C$ is a total dominating set, there exists $b \in T_x \setminus \{a\}$ such that $(x, a)(x, b) \in E(G[K_n])$. It follows that $|T_x| \geq 2$.  

Case 2. Suppose there exists $z \in N_{G}(x) \setminus S$ such that $z \notin N_{G}(y)$ for all $y \in S \setminus \{x\}$.  

Pick $a \in T_x$. If $|T_x| = 1$, then $[C \setminus \{(x, a)\}] \cup \{(z, a)\}$ cannot be a total dominating set in $G[K_n]$, contrary to our assumption that $C$ is a secure total dominating set. Thus, $|T_x| \geq 2$.  

This shows that property (a) holds.  

Next, let $u \in V(G) \setminus S$ with $|N_{G}(u) \cap S| \geq 2$. Pick $v \in V(K_n)$. Since $C$ is a secure total dominating set, there exists $(z, a) \in C$ such that $(u, v)(z, a) \in E(G[K_n])$ and $C^* = [C \setminus \{(z, a)\}] \cup \{(u, v)\}$ is a total dominating set in $G[K_n]$. Suppose $|T_z| = 1$ and let $q \in N_{G}(z) \cap S$. Suppose $|T_q| = 1$, say $t \in T_q$. Since $C^*$ is a total dominating set, there exists $(y, d) \in C^*$ such that $(q, t)(y, d) \in E(G[K_n])$. Clearly, $y \in (S \setminus \{z\}) \cup \{u\}$. Consequently, $qu \in E(G)$ or $q \in N_{G}(y)$, where $y \in S$. This shows that property (b) also holds.  

For the converse, suppose first that (i) holds, that is, $S$ is a secure total dominating set in $G$. Let $(x, y) \in V(G[K_n])$. By assumption, there exists $v \in S$ such that $xv \in E(G)$. Let $a \in T_v$. Then $(x, y)(v, a) \in E(G[K_n])$, showing that $C$ is a total dominating set in $G[K_n]$. Now let $(u, b) \in V(G[K_n]) \setminus C$ and consider the following cases:  

Case 1. Suppose $u \in S$.  

Then, clearly, $b \notin T_u$. Pick $c \in T_u$. Then $(u, b)(u, c) \in E(G[K_n])$ and $[C \setminus \{(u, c)\}] \cup \{(u, b)\}$ is a total dominating set in $G[K_n]$.  

Case 2. Suppose $u \notin S$.  

By assumption, there exists $v \in S$ such that $uv \in E(G)$ and $(S \setminus \{v\}) \cup \{u\}$ is a total dominating set in $G$. Choose $d \in T_y$. Then $(u, b)(v, d) \in E(G[K_n])$ and $[C \setminus \{(v, d)\}] \cup \{(u, b)\}$ is a total dominating set in $G[K_n]$.  

In both cases, $C$ is a secure total dominating set in $G[K_n]$.  

Suppose now that (ii) holds. Let $(x, a) \in V(G[K_n])$. If $x \notin S$, then there exists $z \in S$ such that $xz \in E(G)$. Pick $b \in T_z$. Then $(z, b) \in C$ and $(z, b)(x, a) \in E(G[K_n])$. Suppose $x \in S$. If $|T_x| \geq 2$, then there exists $c \in T_x \setminus \{a\}$ such that $(x, c)(x, c) \in E(G[K_n])$. If $|T_x| = 1$, then by (b), there exists $y \in S$ such that $xy \in E(G)$. Let $d \in T_y$. Then $(y, d) \in C$ and $(y, d)(x, a) \in E(G[K_n])$. Therefore, $C$ is a total dominating set in $G[K_n]$.  

Finally, let \((u, t) \in V(G[K_n]) \setminus C\) and consider the following cases:

**Case 1.** Suppose \(u \in S\).

Pick \(a \in T_u\). Then \((u, a) \in C\), \((u, t)(u, a) \in E(G[K_n])\), and \([C \setminus \{(u, a)\}] \cup \{(u, t)\}\) is a total dominating set in \(G[K_n]\).

**Case 2.** Suppose \(u \notin S\).

If \(|N_G(u) \cap S| = 1\), say \(x \in N_G(u) \cap S\), then \(|T_x| \geq 2\) by (a). Let \(a, b \in T_x\), where \(a \neq b\). Then \((x, a), (x, b) \in C\), \((u, t)(x, a) \in E(G[K_n])\), and \([C \setminus \{(x, a)\}] \cup \{(u, t)\}\) is a total dominating set in \(G[K_n]\).

So suppose \(|N_G(u) \cap S| \geq 2\). Let \(z \in N_G(u) \cap S\) as described in (b). First, suppose that \(|T_z| \geq 2\). Let \(c \in T_z\). Then \((z, c) \in C\), \((u, t)(z, c) \in E(G[K_n])\), and \([C \setminus \{(z, c)\}] \cup \{(u, t)\}\) is a total dominating set in \(G[K_n]\). Suppose now that \(|T_z| = 1\). Let \(k \in T_z\). Then \((z, k) \in C\) and \((z, k)(u, t) \in E(G[K_n])\). Since for each \(q \in N_G(z) \cap S\), \(uq \in E(G)\) or \(|T_q| \geq 2\) or \(q \in N_G(y)\) for some \(y \in S \setminus \{z\}\), it follows that \([C \setminus \{(z, k)\}] \cup \{(u, t)\}\) is a total dominating set in \(G[K_n]\).

Therefore, \(C\) is a secure total dominating set in \(G[K_n]\). \(\square\)

The following is a quick consequence of the above theorem.

**Corollary 3.2** Let \(G\) be a connected graph and \(n \geq 2\). Then

\[\gamma_{st}(G[K_n]) \leq \min\{2\gamma(G), \gamma_{st}(G)\}.\]

**Proof:** Suppose \(S\) is a minimum dominating set in \(G\). Let \(a, b \in V(K_n)\), where \(a \neq b\) and set \(T_x = \{a, b\}\) for every \(x \in S\). Then, by Theorem 3.1, \(C = \bigcup_{x \in S} (\{x\} \times T_x)\) is a secure total dominating set in \(G[K_n]\). Thus, \(\gamma_{st}(G[K_n]) \leq |C| = 2|S| = 2\gamma(G)\). If \(S\) is a minimum secure total dominating set in \(G\), set \(T_x = \{a\}\) for each \(x \in S\). Then again, by Theorem 3.1, \(C = \bigcup_{x \in S} (\{x\} \times T_x)\) is a secure total dominating set in \(G[K_n]\). Thus, \(\gamma_{st}(G[K_n]) \leq |C| = |S| = \gamma_{st}(G)\). Therefore,

\[\gamma_{st}(G[K_n]) \leq \min\{2\gamma(G), \gamma_{st}(G)\}.

The authors were unable to find a graph \(G\) such that strict inequality in Corollary 3.2 holds. Thus, the authors conjectured that the upper bound is actually the exact value of the parameter.

**References**


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