A Laplace Type Problem for a Regular Lattice with Convex-Concave Cell with Obstacles Rhombus and Circular Sections

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Abstract

In the previous papers, [1], [2], [3], [4], [5], [6], [7], [8], [9], the authors studies some Laplace problem for different lattices and different obstacles. In this paper we consider two regular lattices with the cell represented as in figure 1, and we compute the probability that a random segment of constant length intersects a side of lattice. In particular we obtain the probability determinated in the previous work, then the Laplace probability.

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1 Cell with obstacles rhombus and circular sections.

Let $\mathcal{R}_1(a, b, m; \alpha)$ be the regular lattice with the fundamental cell $C_0^{(1)}$ is represented as in figure

\[
\text{fig.1}
\]

where $m < \min(a, b)$ and $\alpha \leq \frac{\pi}{2}$ an angle. The four obstacles are two rhombus and two circular sections.

We have:

\[
\text{area } C_0^{(1)} = 2ab\sin\alpha - \frac{m^2 \sin\alpha}{2} - \frac{\pi m^2}{8}. \tag{1}
\]

Considering a random segment $s$ of constant length $l$ with $l < \min(a - m, b - \frac{m}{2})$ and we compute the probability $P_{\text{int}}^{(1)}$ that this segment intersects a side of lattice, therefore the probability that the segment $s$ intersects the side of $C_0^{(1)}$.

We consider a segment $s$ of random position, determinated by his middle point $0$ and by angle $\varphi$ that it forms with the side $CD$ of $C_0^{(1)}$.

To compute the probability $P_{\text{int}}^{(1)}$ we consider the limiting positions of the segment $s$ for a determinated value of $\varphi$ let $\hat{C}_0^{(1)}(\varphi)$ be the determinated as in fig. 2:
From here we can write:

\[
\text{area} \hat{C}_{0}^{(1)} (\varphi) = \text{area} C_{0}^{(1)} - \\
[\text{area} d_{1} (\varphi) + \text{area} d_{2} (\varphi) + ... + \text{area} d_{11} (\varphi)].
\]  

(2)

\[
\text{area} d_{1} (\varphi) + \text{area} d_{2} (\varphi) = \frac{l^{2} \sin \varphi \sin (\alpha - \varphi)}{2 \sin \alpha} - \frac{m^{2} \sin \alpha}{4},
\]

\[
\text{area} d_{3} (\varphi) = \left( a - \frac{m}{2} - \frac{l \sin \varphi}{\sin \alpha} \right) \frac{l}{2} \sin (\alpha - \varphi),
\]

\[
\text{area} d_{5} (\varphi) = \left( b - \frac{m}{2} \right) \frac{l}{2} \sin \varphi,
\]

\[
\text{area} d_{6} (\varphi) = \left[ b - \frac{l \sin (\alpha - \varphi)}{\sin \alpha} \right] \frac{l}{2} \sin (2 \alpha - \varphi),
\]

\[
\text{area} d_{9} (\varphi) = \frac{ml}{2} \cos \frac{\alpha}{2} \sin \left( \frac{\alpha}{2} + \varphi \right) - \frac{m^{2}}{8} \sin \alpha,
\]

\[
\text{area} d_{10} (\varphi) = \left( b - \frac{m}{2} \right) \frac{l}{2} \sin (2 \alpha - \varphi),
\]

\[
\text{area} d_{11} (\varphi) = \left[ b - \frac{l \sin (\alpha - \varphi)}{\sin \alpha} \right] \frac{l}{2} \sin \varphi,
\]
\[
\text{area}_{4} (\varphi) = \frac{ml}{2} \sin \frac{\alpha}{2} \cos \left( \frac{\alpha}{2} - \varphi \right) - \frac{m^2 (\alpha - \sin \alpha)}{8},
\]
\[
\text{area}_{7} (\varphi) = \frac{l^2 \sin (2\alpha - \varphi) \sin (\alpha - \varphi)}{2 \sin \alpha} - \frac{am^2}{8},
\]
\[
\text{area}_{8} (\varphi) = \text{area}_{c} (\varphi) = \left[ a - \frac{m}{2} - \frac{l \sin (2\alpha - \varphi)}{\sin \alpha} \right] \cdot \frac{l}{2} \sin (\alpha - \varphi). \tag{3}
\]

We have that
\[
\text{area}_{c}^{(1)} (\varphi) = \text{area}_{c}^{(1)} - \left\{ \frac{l^2 \sin \varphi \sin (\alpha - \varphi)}{2 \sin \alpha} - \frac{m^2 \sin \alpha}{4} + \right. \\
\left. + \left( a - \frac{m}{2} - \frac{l \sin \varphi}{\sin \alpha} \right) \frac{l}{2} \sin (\alpha - \varphi) + \left( b - \frac{m}{2} \right) \frac{l}{2} \sin \varphi + \\
+ \left[ b - \frac{l \sin (\alpha - \varphi)}{\sin \alpha} \right] \cdot \frac{l}{2} \sin (2\alpha - \varphi) + \frac{ml}{2} \cos \frac{\alpha}{2} \sin \left( \frac{\alpha}{2} + \varphi \right) + \\
- \frac{m^2 \sin \alpha}{8} + \left( b - \frac{m}{2} \right) \frac{l}{2} \sin (2\alpha - \varphi) + \left[ b - \frac{l \sin (\alpha - \varphi)}{\sin \alpha} \right] \frac{l}{2} \sin \varphi + \\
\frac{ml}{2} \sin \frac{\alpha}{2} \cos \left( \frac{\alpha}{2} - \varphi \right) - \frac{m^2 (\alpha - \sin \alpha)}{8} + \\
+ \frac{l^2 \sin (2\alpha - \varphi) \sin (\alpha - \varphi)}{2 \sin \alpha} - \frac{am^2}{8} + \\
+ \left[ a - \frac{m}{2} - \frac{l \sin (2\alpha - \varphi)}{\sin \alpha} \right] \cdot \frac{l}{2} \sin (\alpha - \varphi) \right\} = \\
\text{area}_{c}^{(1)} - \left\{ \left[ 2a \sin \alpha + \left( 2b - \frac{m}{2} \right) \sin 2\alpha \right] \frac{l}{2} \cos \varphi + \\
+ \left[ 2b + \frac{m}{2} - (2a - m) \cos \alpha - \left( 2b - \frac{m}{2} \right) \cos 2\alpha \right] \frac{l}{2} \sin \varphi + \\
- l^2 \cdot \frac{\sin \varphi \sin (\alpha - \varphi)}{\sin \alpha} - \frac{m^2}{8} (\alpha + 2 \sin \alpha) \right\}. \tag{4}
\]
Denoting with $M_1$ the set of all segments $s$ which have the center point in $C_0^{(1)}$ and with $N_1$ the set of all segments $s$ completely contained in $C_0^{(1)}$, we have that:

$$P_{int}^{(1)} = 1 - \frac{\mu(N_1)}{\mu(M_1)},$$

and

$$\mu(M_1) = \int_0^\alpha d\varphi \iint_{\{(x,y)\in C_0^{(1)}\}} dx dy = \int_0^\alpha \left[ \text{area}C_0^{(1)} \right] d\varphi = \alpha \text{area}C_0^{(1)}$$

we have that

$$\mu(N_1) = \int_0^\alpha d\varphi \iint_{\{(x,y)\in \hat{C}_0^{(1)}(\varphi)\}} dx dy = \int_0^\alpha \left[ \text{area}C_0^{(1)}(\varphi) \right] d\varphi =$$

$$\alpha \text{area}C_0^{(1)} - \left\{ \left[ 2a \sin \alpha + \left( 2b - \frac{m}{2} \right) \sin 2\alpha \right] \frac{l}{2} \sin \varphi -$$

$$- \left[ 2b + \frac{m}{2} - (2a - m) \cos \alpha - \left( 2b - \frac{m}{2} \right) \cos 2\alpha \right] \frac{l}{2} \sin \varphi +$$

$$+ \frac{l^2}{2 \sin \alpha} \left[ \frac{\sin (\alpha - 2\varphi)}{2} \cos \alpha - \frac{m^2 \varphi}{2} (\pi - \sin \alpha) \right] \right\}_0^\alpha =$$

$$\alpha \text{area}C_0^{(1)} - \left\{ \left[ 2a \sin \alpha + \left( 2b - \frac{m}{2} \right) \sin 2\alpha \right] \frac{l}{2} \sin \alpha +$$

$$\left[ 2b + \frac{m}{2} - (2a - m) \cos \alpha - \left( 2b - \frac{m}{2} \right) \cos 2\alpha \right] \frac{l}{2} (1 - \cos \alpha)$$

$$- \frac{l^2}{2} (1 - \alpha \cot \alpha) - \frac{m^2 \alpha}{2} (\pi - \sin \alpha) \right\}.$$  

The relations (5), (6), and (7) give us:

$$P_{int}^{(1)} = \frac{1}{\alpha \left( 2ab \sin \alpha - \frac{m^2 \sin \alpha}{2} - \frac{\pi m^2}{8} \right)},$$

$$\left\{ \left[ 2a \sin \alpha + \left( 2b - \frac{m}{2} \right) \sin 2\alpha \right] \frac{l}{2} \sin \alpha +$$

$$\left[ 2b + \frac{m}{2} - (2a - m) \cos \alpha - \left( 2b - \frac{m}{2} \right) \cos 2\alpha \right] \frac{l}{2} (1 - \cos \alpha)$$
\[ -\frac{l^2}{2} \left( 1 - \alpha ctg\alpha \right) - \frac{m^2\alpha}{2} \left( \pi - \sin\alpha \right) \right\}. \quad (8) \]

For \( \alpha = \frac{\pi}{2} \) and \( m = 0 \), the fundamental cell become a rectangle with side \( a \) and \( 2b \) and the probability (8) become the Laplace probability:

\[ P = \frac{2(a + 2b)l - l^2}{2\pi ab}. \]

References


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