On the Numerical Ranges of a Tridiagonal Operators

A. Abdollahi

Department of Mathematics, Shiraz University, Shiraz, Iran

Abstract

Assume that \( A \) is a tridiagonal operator on \( \ell^2(\mathbb{N}) \), defined by \( Ae_j = e_{j-1} + (-1)^je_{j+1}, j = 1, 2, \cdots \), where \( \{e_1, e_2, e_3, \cdots \} \) is the standard orthonormal basis for \( \ell^2(\mathbb{N}) \). In 2011, M. Chien and H. Nakazato have studied the numerical range of these type of tridiagonal operators. In this paper we give another prove of the results.

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1 Introduction

For a bounded linear operator \( T \) on a Hilbert space \( \mathcal{H} \), the numerical range \( W(T) \) is the image of the unit sphere of \( \mathcal{H} \) under the quadratic form \( x \mapsto < Tx, x > \) associated with the operator. More precisely,

\[
W(T) = \{ < Tx, x > : x \in \mathcal{H}, \|x\| = 1 \}.
\]

A complete survey on numerical range can be found in the books by F. Bonsall and J. Duncan [1], [2] and the book by K. E. Gustafon and K. M. Rao [5], and we refer the reader to these books for general information and background.
Consider an operator in infinite matrix form

\[
A(\infty, -1) = \begin{pmatrix}
0 & 1 \\
-1 & 0 & 1 \\
1 & 0 & 1 \\
-1 & 0 & 1 \\
\vdots & \ddots & \ddots \\
\end{pmatrix},
\]

and a finite matrix

\[
A(n, -1) = \begin{pmatrix}
0 & 1 \\
-1 & 0 & 1 \\
1 & 0 & 1 \\
\vdots & \ddots & \ddots \\
(-1)^{n-2} & 0 & 1 \\
(-1)^{n-1} & 0 \\
\end{pmatrix},
\]

In [3], M. T. Chien and H. Nakazato have determined the numerical range of \(A(\infty, -1)\) and \(A(2n + 1, -1)\), for each nonnegative integer \(n\). Here we find the results with another method.

For the study of numerical ranges of finite matrices, the matrix-theoretic properties can be exploited to yield special tools which are not available for general operators. One important way to yield \(\partial W(A)\) is the Kippenhahn’s result that the numerical range of \(A\) coincides with the convex hull of the real points of the dual curve of \(\det(xReA + yImA + zI) = 0\). On the other hand, a parametric representation of the boundary \(W(A)\) can also be obtained from the largest eigenvalue of \(Re(e^{-i\theta}A)\) yielding useful information on \(W(A)\).

For any \(n \times n\) matrix \(A\), let \(\lambda(\theta)\) denote the maximum eigenvalue of \(Re(e^{-i\theta}A)\). It is well known that \(\lambda(\theta)\) is an analytic function of \(\theta\) (possibly except for some isolated points), and a unit vector in \(\mathbb{C}^n\) is such that \(\langle Ax, x \rangle\) belong to \(\partial W(A) \cap L_\theta\) if and only if \(Re(e^{-i\theta}A)x = \lambda(\theta)x\). Also \(\partial W(A)\) admits a parametric representation

\[
x(\theta) = \lambda(\theta) \cos(\theta) - \lambda'(\theta) \sin(\theta);
y(\theta) = \lambda(\theta) \sin(\theta) + \lambda'(\theta) \cos(\theta),
\]

(again, with possible exception of finitely many points).

So it is enough to compute the \(\lambda(\theta)\) (see [4]). Put \(M_n = Re(e^{-i\theta}A_n) - \lambda I_n\) and \(P_n(\lambda) = det M_n\). By a simple computation we get the following recursion formula:

\[
P_n(\lambda) = \lambda P_{n-1}(\lambda) - \cos^2((n - 1)\frac{\pi}{2} - \theta)P_{n-2}(\lambda)
\]
with initial conditions $P_0(\lambda) = 1$ and $P_0(\lambda) = \lambda$.

For each $n \geq 0$, define $H_n = P_{2n+1}$. Then these polynomials satisfy the following recursion formula:

$$H_n(\lambda) = (\lambda^2 - 1)H_{n-1}(\lambda) - z^2 H_{n-2}(\lambda), n \geq 2$$

with initial conditions $H_0(\lambda) = -\lambda$ and $H_1(\lambda) = -\lambda(\lambda^2 - 1)$, where $z = \cos(\theta) \sin(\theta)$. Hence

$$H_n = \det \begin{pmatrix} -\lambda & \lambda^2 - 1 & z & z & \ldots & z & \lambda^2 - 1 \\ z & \lambda^2 - 1 & \lambda^2 - 1 & z & \ldots & z & \lambda^2 - 1 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \lambda^2 - 1 & z & \lambda^2 - 1 & z & \lambda^2 - 1 \\ & & & & \lambda^2 - 1 & z & \lambda^2 - 1 \\ & & & & & \lambda^2 - 1 & z & \lambda^2 - 1 \\ & & & & & & \lambda^2 - 1 & z & \ldots & \ldots & \lambda^2 - 1 \\ & & & & & & & \lambda^2 - 1 & z & \ldots & \ldots & \ldots & \lambda^2 - 1 \end{pmatrix}$$

$$= -\lambda \det \begin{pmatrix} \lambda^2 - 1 & z & z & z & \ldots & z & \lambda^2 - 1 \\ z & \lambda^2 - 1 & \lambda^2 - 1 & z & \ldots & z & \lambda^2 - 1 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \lambda^2 - 1 & z & \lambda^2 - 1 & z & \lambda^2 - 1 \\ & & & \lambda^2 - 1 & z & \lambda^2 - 1 & z & \lambda^2 - 1 \\ & & & & \lambda^2 - 1 & z & \lambda^2 - 1 & z & \lambda^2 - 1 \\ & & & & & \lambda^2 - 1 & z & \lambda^2 - 1 & z & \lambda^2 - 1 & \lambda^2 - 1 \\ & & & & & & \lambda^2 - 1 & z & \ldots & \ldots & \ldots & \lambda^2 - 1 \end{pmatrix}$$

$$= (-\lambda) \prod_{j=1}^{n}(\lambda^2 - 1 - 2z \cos(\frac{j\pi}{n+1}))$$

Therefore the eigenvalues of the matrix $M_{2n+1}$ are satisfy

$$\lambda_0 = 0, \text{ and } \lambda_j^2 = 1 + \sin(2\theta) \cos(\frac{j\pi}{n+1}), j = 1, 2, \ldots, n.$$ 

In the case that $0 < \theta < \frac{\pi}{2}, \sin(2\theta) > 0$ and so the maximum eigenvalue attained for $j = 1$, and it follows that

$$\lambda(\theta) = \sqrt{1 + \sin(2\theta) \cos(\frac{\pi}{n+1})}.$$ 

In the case that $\frac{\pi}{2} < \theta < \pi, \sin(2\theta) < 0$ and so the maximum eigenvalue attained for $j = n$, and it follows that

$$\lambda(\theta) = \sqrt{1 + \sin(2(\theta - \frac{\pi}{2})) \cos(\frac{\pi}{n+1})} = \sqrt{1 + \sin(2(\theta - \frac{\pi}{2}))} \cos(\frac{\pi}{n+1}) = \lambda(\theta - \frac{\pi}{2}).$$
By similar way it is easy to show that for each $\theta, \lambda(\theta) = \lambda(\theta - \frac{\pi}{2})$. So we have the following proposition as a result of the above arguments.

**Proposition 1.1.** The numerical range of the matrix $M_{2n+1}$ is rotation invariant under the rotation by $\frac{\pi}{2}$.

Now by using (1.1), for $0 < \theta < \frac{\pi}{2}$, we have the following parametric equation for a part of the boundary of $W(A(2n+1,-1))$,

$$
\begin{align*}
x(\theta) &= \frac{\cos(\theta) + \sin(\theta) \cos(\frac{\pi}{n+1})}{\lambda(\theta)}; \\
y(\theta) &= \frac{\sin(\theta) + \cos(\theta) \cos(\frac{\pi}{n+1})}{\lambda(\theta)},
\end{align*}
$$

(1.2)

which is a part of the ellipse with an equation

$$x^2 - 2xy \cos(\frac{\pi}{n+1}) + y^2 = \sin(\frac{\pi}{n+1}).$$

Also by a simple computation, we have $\lim_{\theta \to 0^-} y(\theta) = \cos(\frac{\pi}{n+1})$ and $\lim_{\theta \to 0^+} y(\theta) = 1$ and $\lim_{\theta \to \frac{\pi}{2}^-} x(\theta) = \cos(\frac{\pi}{n+1})$ and $\lim_{\theta \to \frac{\pi}{2}^-} x(\theta) = 1$. Hence, by Proposition 1.1, $\lim_{\theta \to 0^-} y(\theta) = -\cos(\frac{\pi}{n+1})$ and $\lim_{\theta \to 0^-} x(\theta) = 1$. For $\theta = 0$, the support line $L_0$ is the line $x = 1$ and so the segment $\{(1, y) : -\cos(\frac{\pi}{n+1}) \leq y \leq \cos(\frac{\pi}{n+1})\}$ is a flat portion of the boundary of $W(A(2n+1,-1))$. Therefore we can state the following Theorem:

**Theorem 1.2.**

a) the numerical range of $A(\infty, -1)$ is the square with vertices at $(1, 1), (-1, 1), (-1, -1)$ and $(1, -1)$.

b) for $m = 2n + 1$, the numerical range $W(A(m, -1))$ is the convex hull of the two ellipses, and has 4 flat portions on the boundary of $W(A(m, -1))$, which are also on the edges of the square.

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**References**


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