Baire One-Stieltjes Integration

Karlo S. Orge

Department of Mathematics and Statistics
College of Science and Mathematics
Mindanao State University - Iligan Institute of Technology
Iligan City 9200, Philippines

Julius V. Benitez

Department of Mathematics and Statistics
College of Science and Mathematics
Mindanao State University - Iligan Institute of Technology
Iligan City 9200, Philippines

Abstract

This study introduces the Baire One-Stieltjes integral. The integral is shown to possess the desired elementary integral properties, including Cauchy Criterion and Henstock Lemma. It is known that every Riemann-Stieltjes integrable function is Baire One-Stieltjes integrable, while every Baire One-Stieltjes integrable function is Henstock-Stieltjes integrable. Between the Riemann-Stieltjes and the Baire One-Stieltjes, strict inclusion is exhibited in the space of regulated functions.

We also formulate convergence theorems for this Stieltjes integral such as Uniform Convergence Theorem, Equi-integrability Convergence Theorem and Monotone Convergence Theorem. Also, some results leading to Riesz Representation Theorem are presented for this Stiletjes integral.

\footnote{Research supported by the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP), Philippines}
1 Introduction

Stieltjes integrals are extensions of ordinary integrals in the sense that the length $v - u$ of the sub-interval $[u, v]$ used in defining the integral is replaced by $g(v) - g(u)$, for some real-valued function $g$. The first Stieltjes integral, the Riemann-Stieltjes integral, was introduced by T.J. Stieltjes in 1894. F. Riesz, who brought the integral into attention in 1914 through the Riesz Representation Theorem, shows that the any linear functional on the space of continuous functions can be expressed using this integral. Thereafter, the Riemann-Stieltjes integral is thoroughly studied, as shown in [9] and other researchers formulated Stieltjes integrals for other integrals, such as the Lebesgue integral [6] and the Henstock integral [7]. The Stieltjes integrals were used extensively to prove some concepts in functional analysis [5] and probability theory [6].

On the other hand, it is well-known that the pointwise limit of continuous functions on $[a, b]$ is not necessarily continuous on $[a, b]$. Such functions which are pointwise limits of continuous functions are commonly known as Baire One functions. Recently, Lee, Tang, Zhao in [8] introduced an alternative definition of Baire One functions in terms of the usual $\varepsilon$-$\delta$ format. With this $\varepsilon$-$\delta$ definition, Caroline Lee Su Yin in [1] developed the Baire One integral. With this new integral, she proved some basic properties and established the relationship of this integral with the known Riemann and Henstock integrals.

With the integral introduced by Lee in [1], this paper attempts to construct a Stieltjes version of such integral and study its properties.

2 Preliminaries

Let $D = \{x_0, x_1, x_2, \ldots, x_n\}$ be a division of a closed and bounded interval $[a, b]$. By a tagged division $\mathcal{D}$ of $[a, b]$, we mean a finite set of interval-point pairs

$$\mathcal{D} = \{([x_0, x_1], \xi_1), ([x_1, x_2], \xi_2), \ldots, ([x_{n-1}, x_n], \xi_n)\},$$

such that for each $i = 1, 2, \ldots, n$, $\xi_i \in [x_{i-1}, x_i]$. Each $\xi_i$ is called a tag in $\mathcal{D}$ associated with the interval $[x_{i-1}, x_i]$. Any non-empty subset of a tagged division of $[a, b]$ is called a tagged partial division of $[a, b]$.

By a gauge on $[a, b]$, we mean a function $\delta: [a, b] \to \mathbb{R}^+$. If $\delta$ is a gauge on $[a, b]$, then a tagged division or a tagged partial division $\mathcal{D} = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n$
of \([a,b]\) is said to be \(\delta\)-fine if for all \(i = 1, 2, \ldots, n\), we have \([x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))\).

Let \(f\) and \(g\) be real-valued functions on \([a, b]\) and let \(D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n\) be a tagged division or a tagged partial division of \([a, b]\). The Riemann sum of \(f\) with respect to \(g\) corresponding to \(D\), denoted by \(S(f, g, D)\), is the finite sum

\[
S(f, g, D) = \sum_{i=1}^n f(\xi_i)[g(x_i) - g(x_{i-1})].
\]

### 3 The Baire One-Stieltjes Integral

When Lee in [1] constructed the Baire One integral, the concept of “compatible” tagged divisions is used in the definition. We present such concept in the following definition.

**Definition 3.1.** [1] Let \(\delta\) be a gauge on \([a, b]\) and suppose that \(D = \{([x_{i-1}, x_i], \xi_i)\}_{i=1}^n\) and \(D^* = \{([x_{i-1}, x_i], \eta_i)\}_{i=1}^n\) be tagged divisions of \([a, b]\). We say that \(D^*\) is \(D\)-compatible, written \(D^* \sim D\), if \(|\eta_i - \xi_i| < \delta(\eta_i)\) for all \(i = 1, 2, \ldots, n\).

Observe that if \(D\) is a \(\delta\)-fine tagged division of \([a, b]\), a \(D\)-compatible tagged division may not be \(\delta\)-fine.

Let \(c \in (a, b)\), \(\delta_1\) a gauge on \([a, c]\) and \(\delta_2\) a gauge on \([c, b]\). It is known that for the gauge \(\delta\) on \([a, b]\) defined by

\[
\delta(x) = \begin{cases} 
\min\{\delta_1(x), c - x\}, & \text{if } x \in [a, c), \\
\min\{\delta_1(x), \delta_2(x)\}, & \text{if } x = c, \\
\min\{\delta_2(x), x - c\}, & \text{if } x \in (c, b], 
\end{cases}
\]

(1)

c is a tag in every \(\delta\)-fine tagged division \(D\) of \([a, b]\). By the definition of a \(D\)-compatible division, it can be shown that if \(D^* \sim D\) and \((u, v], c) \in D\), then \((u, v], c) \in D^*\).

**Definition 3.2.** A function \(f: [a, b] \to \mathbb{R}\) is said to be Baire One-Stieltjes integrable with respect to a function \(g: [a, b] \to \mathbb{R}\) on \([a, b]\) to \(A \in \mathbb{R}\) if for every \(\epsilon > 0\), there exists a gauge \(\delta\) on \([a, b]\) such that for all \(\delta\)-fine tagged division \(D\) of \([a, b]\) and for all \(D^* \sim D\), we have

\[
|S(f, g, D^*) - A| < \epsilon.
\]

We call \(A\) as the Baire One-Stieltjes integral of \(f\) with respect to \(g\) on \([a, b]\) and write

\[
A = (\mathcal{B}) \int_a^b f \, dg.
\]
Proposition 3.3. Let $f, g : [a, b] \to \mathbb{R}$. If $f$ is Baire One-Stieltjes integrable with respect to $g$ on $[a, b]$, then its integral is unique.

Proof: Suppose that $f$ is Baire One-Stieltjes integrable with respect to $g$ on $[a, b]$ to $A$ and $B$. Let $\epsilon > 0$. Then there exists a gauge $\delta_1$ on $[a, b]$ such that for all $\delta_1$-fine tagged division $\mathcal{D}_1$ and for all $\mathcal{D}_1^* \sim \mathcal{D}_1$, we have
\[ |S(f, g, \mathcal{D}_1^*) - A| < \frac{\epsilon}{2}. \]
Similarly, there exists a gauge $\delta_2$ on $[a, b]$ such that for all $\delta_2$-fine tagged division $\mathcal{D}_2$ and for all $\mathcal{D}_2^* \sim \mathcal{D}_2$, we have
\[ |S(f, g, \mathcal{D}_2^*) - B| < \frac{\epsilon}{2}. \]
Let $\delta = \min\{\delta_1, \delta_2\}$ on $[a, b]$. Then $\delta$ is a gauge on $[a, b]$. Fix a $\delta$-fine tagged division $\mathcal{D}$ of $[a, b]$. Then $\mathcal{D}$ is both $\delta_1$-fine and $\delta_2$-fine, so that given a fixed tagged division $\mathcal{D}^*$ of $[a, b]$ with $\mathcal{D}^* \sim \mathcal{D}$, we have
\[ |S(f, g, \mathcal{D}^*) - A| < \frac{\epsilon}{2} \quad \text{and} \quad |S(f, g, \mathcal{D}^*) - B| < \frac{\epsilon}{2}. \]
It follows from the above inequalities that $|A - B| < \epsilon$. That is, $A = B$. \qed

For real-valued functions, let $\mathcal{B}([a, b])$ (resp. $\mathcal{R}([a, b])$ and $\mathcal{H}([a, b])$) be the collection of pairs $(f, g)$ of real-valued functions on $[a, b]$ such that $f$ is Baire One-Stieltjes (resp. Riemann-Stieltjes and Henstock-Stieltjes) integrable with respect to $g : [a, b] \to \mathbb{R}$ on $[a, b]$. Then we have the following.

Proposition 3.4. $\mathcal{R}([a, b]) \subseteq \mathcal{B}([a, b]) \subseteq \mathcal{H}([a, b])$.

It is still unknown whether $\mathcal{H}([a, b]) \subseteq \mathcal{B}([a, b])$. However, one condition for a Henstock-Stieltjes integrable function with respect to $g$ on $[a, b]$ to be Baire One-Stieltjes integrable with respect to $g$ on $[a, b]$ is given by the next theorem.

Theorem 3.5. Let $(f, g) \in \mathcal{H}([a, b])$. If $f$ is a Baire One function and $g$ is of bounded variation, then $(f, g) \in \mathcal{B}([a, b])$.

4 Simple Properties

The following theorem follows directly from the definition of the Baire One-Stieltjes integral.

Theorem 4.1. Suppose that $f$ and $f^*$ are Baire One-Stieltjes integrable with respect to $g$ on $[a, b]$ and $f$ is Baire One-Stieltjes integrable with respect to $g^*$ on $[a, b]$. Let $\alpha \in \mathbb{R}$. Then
(i) \( f + f^* \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b]\), and
\[
(\mathcal{B}) \int_a^b (f + f^*) \, dg = (\mathcal{B}) \int_a^b f \, dg + (\mathcal{B}) \int_a^b f^* \, dg.
\]

(ii) \( \alpha f \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b]\), and
\[
(\mathcal{B}) \int_a^b \alpha f \, dg = \alpha \cdot (\mathcal{B}) \int_a^b f \, dg.
\]

(iii) \( f \) is Baire One-Stieltjes integrable with respect to \( g + g^* \) on \([a, b]\), and
\[
(\mathcal{B}) \int_a^b f \, d(g + g^*) = (\mathcal{B}) \int_a^b f \, dg + (\mathcal{B}) \int_a^b f \, dg^*.
\]

(iv) \( f \) is Baire One-Stieltjes integrable with respect to \( \alpha g \) on \([a, b]\), and
\[
(\mathcal{B}) \int_a^b f \, d(\alpha g) = \alpha \cdot (\mathcal{B}) \int_a^b f \, dg.
\]

**Theorem 4.2.** Let \( f, g : [a, b] \to \mathbb{R} \) and \( c \in (a, b) \). If \( f \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, c]\) and \([c, b]\), then \( f \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b]\), and
\[
(\mathcal{B}) \int_a^b f \, dg = (\mathcal{B}) \int_a^c f \, dg + (\mathcal{B}) \int_c^b f \, dg.
\]

**Proof:** Let \( \epsilon > 0 \), \( A = (\mathcal{B}) \int_a^c f \, dg \) and \( B = (\mathcal{B}) \int_c^b f \, dg \). By hypothesis, there exists a gauge \( \delta_1 \) on \([a, c]\) such that for all \( \delta_1 \)-fine tagged divisions \( \mathcal{D} \) on \([a, c]\) and \( \mathcal{D}^* \sim \mathcal{D} \), we have
\[
|S(f, g, \mathcal{D}^*) - A| < \frac{\epsilon}{2}.
\]
Also, there exists a gauge \( \delta_2 \) on \([c, b]\) such that for all \( \delta_2 \)-fine tagged divisions \( \mathcal{D} \) on \([c, b]\) and \( \mathcal{D}^* \sim \mathcal{D} \), we have
\[
|S(f, g, \mathcal{D}^*) - B| < \frac{\epsilon}{2}.
\]
Define the gauge \( \delta \) on \([a, b]\) as in equation (1). Let \( \mathcal{D} \) be a \( \delta \)-fine tagged division of \([a, b]\) and \( \mathcal{D}^* \sim \mathcal{D} \). Then \( c \) is a tag in \( \mathcal{D} \) and \( \mathcal{D} \), and so there exists \(([u, v], c) \in \mathcal{D} \cap \mathcal{D}^* \). We can assume that \( c \) is a division point in \( \mathcal{D} \) and \( \mathcal{D}^* \);
otherwise, if \( u < c < v \), replace \(([u, v], c)\) in \( \mathcal{D} \) and \( \mathcal{D}^* \) by the pairs \(([u, c], c)\) and \(([c, v], c)\). Let

\[
\mathcal{D}_1 = \{([u, v], \xi) \in \mathcal{D} : [u, v] \subseteq [a, c]\}, \\
\mathcal{D}_2 = \{([u, v], \xi) \in \mathcal{D} : [u, v] \subseteq [c, b]\}, \\
\mathcal{D}_1^* = \{([u, v], \eta) \in \mathcal{D}^* : [u, v] \subseteq [a, c]\}, \\
\mathcal{D}_2^* = \{([u, v], \eta) \in \mathcal{D}^* : [u, v] \subseteq [c, b]\}.
\]

Then \( \mathcal{D}_1 \) is a \( \delta_1 \)-fine tagged division of \([a, c]\) and \( \mathcal{D}_2 \) is a \( \delta_2 \)-fine tagged division of \([c, b]\). Moreover, \( \mathcal{D}_1^* \sim \mathcal{D}_1 \), \( \mathcal{D}_2^* \sim \mathcal{D}_2 \), \( \mathcal{D}^* = \mathcal{D}_1^* \cup \mathcal{D}_2^* \), and

\[ S(f, g, \mathcal{D}^*) = S(f, g, \mathcal{D}_1^*) + S(f, g, \mathcal{D}_2^*). \]

Hence,

\[ |S(f, g, \mathcal{D}^*) - (A + B)| \leq |S(f, g, \mathcal{D}_1^*) - A| + |S(f, g, \mathcal{D}_2^*) - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

Therefore, \( f \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b]\), and

\[ (B) \int_a^b f \, dg = A + B = (B) \int_a^c f \, dg + (B) \int_c^b f \, dg. \]

The following two theorems can be easily verified and so the proofs are omitted.

**Theorem 4.3. (Cauchy Criterion)** Let \( f, g : [a, b] \to \mathbb{R} \). Then \( f \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b]\) if and only if for all \( \epsilon > 0 \), there exists a gauge \( \delta \) on \([a, b]\) such that for all \( \delta \)-fine tagged divisions \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) of \([a, b]\), and for all \( \mathcal{D}_1^* \sim \mathcal{D}_1 \) and \( \mathcal{D}_2^* \sim \mathcal{D}_2 \), we have

\[ |S(f, g, \mathcal{D}_1^*) - S(f, g, \mathcal{D}_2^*)| < \epsilon. \]

**Theorem 4.4.** Let \( f, g : [a, b] \to \mathbb{R} \). If \( f \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b]\), then \( f \) is Baire One-Stieltjes integrable with respect to \( g \) on every closed sub-interval \([c, d]\) of \([a, b]\).

5 An Existence Theorem and Convergence Theorems

**Definition 5.1.** [[3]] Let \( g : [a, b] \to \mathbb{R} \) be a function. The variation of \( g \) with respect to a division \( D = \{x_0, x_1, x_2, \ldots, x_n\} \) of \([a, b]\) is defined by

\[ v(g, D) = \sum_{i=0}^{n} |g(x_i) - g(x_{i-1})|. \]
The variation of $g$ over $[a,b]$ is defined by

$$V(g, [a,b]) = \sup \{v(g, D) : D \text{ is a division of } [a,b]\}.$$ 

If $V(g, [a,b]) < \infty$, then we say that $g$ is of bounded variation on $[a,b]$.

**Definition 5.2.** [2] A function $f : [a, b] \to \mathbb{R}$ is said to be regulated if for every $\xi \in [a,b)$, the right-hand limit $f(\xi^+)$ exists, and for every $\xi \in (a,b]$, the left-hand limit $f(\xi^-)$ exists.

Since the class of regulated functions contains the class of continuous functions and the class of functions of bounded variation, this class will be investigated for the existence of the Baire One-Stieltjes integral.

**Theorem 5.3.** Let $f, g : [a, b] \to \mathbb{R}$ be regulated functions. If $g$ is of bounded variation and $f$ and $g$ do not have common one-sided discontinuities, then $f$ is Baire-One Stieltjes integrable with respect to $g$.

The following convergence theorems can be proved in a similar argument as in the corresponding convergence theorems for other Stieltjes integrals.

**Theorem 5.4.** (Uniform Convergence Theorem) Let $g$ be a real-valued function on $[a, b]$ which is of bounded variation and $\{f_n\}_{n=1}^\infty$ be a sequence of real-valued functions on $[a, b]$ such that $f_n$ is Baire One-Stieltjes integrable with respect to $g$ on $[a, b]$, for all $n \in \mathbb{N}$. If $f$ is a real-valued function on $[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$, then $f$ is Baire One-Stieltjes integrable with respect to $g$ on $[a, b]$, and

$$\left(\mathcal{B}\right) \int_a^b f \, dg = \lim_{n \to \infty} \left(\mathcal{B}\right) \int_a^b f_n \, dg.$$ 

**Theorem 5.5.** (Equi-integrability Convergence Theorem) Let $g$ be a real-valued function on $[a, b]$ which is of bounded variation, and suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of real-valued functions on $[a, b]$ which is equi-integrable with respect to $g$. If $f$ is a real-valued function on $[a, b]$ such that $f_n \to f$ pointwisely on $[a, b]$, then $f$ is Baire One-Stieltjes integrable with respect to $g$ on $[a, b]$, and

$$\left(\mathcal{B}\right) \int_a^b f \, dg = \lim_{n \to \infty} \left(\mathcal{B}\right) \int_a^b f_n \, dg.$$ 

**Theorem 5.6.** (Monotone Convergence Theorem) Let $g$ be a real-valued increasing function on $[a, b]$ and suppose that $\{f_n\}_{n=1}^\infty$ is an increasing sequence of Baire One-Stieltjes integrable functions with respect to $g$ on $[a, b]$ which converges pointwisely to a real-valued function $f$ on $[a, b]$. If

$$\sup \left\{ \left(\mathcal{B}\right) \int_a^b f_n \, dg : n \in \mathbb{N} \right\} < \infty,$$
then \( f \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b] \), and

\[
(\mathcal{B}) \int_a^b f \, dg = \lim_{n \to \infty} (\mathcal{B}) \int_a^b f_n \, dg.
\]

6 Results Related to the Riesz Representation Theorem

The Riesz Representation Theorem is a theorem which expresses any linear functional on the space of continuous functions in terms of a Riemann-Stieltjes integral. This section formulates some results which lead to an analogue of Riesz Representation Theorem in terms of the Baire One-Stieltjes integral.

We will show that Theorem 5.3 remains true even if \( f \) and \( g \) have common one-sided discontinuities. First, we will show an example that such Stieltjes integral exists.

**Proposition 6.1.** Let \( c \in [a, b] \) and \( \alpha, \theta \in \mathbb{R} \). Define \( \psi_c : [a, b] \to \mathbb{R} \) by the following:

\[
\psi_c(x) = \begin{cases} 
\alpha, & \text{if } x = c, \\
\theta, & \text{if } x \neq c.
\end{cases}
\]

If \( g : [a, b] \to \mathbb{R} \) is regulated, then \( \psi_c \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b] \) and

\[
(\mathcal{B}) \int_a^b \psi_c \, dg = (\alpha - \theta)[g(c^+) - g(c^-)] + \theta[g(b) - g(a)].
\]

(For convenience, we let \( g(a^-) = g(a) \) and \( g(b^+) = g(b) \).)

Using the above proposition and Theorem 4.2, we obtain a more general result.

**Theorem 6.2.** Let \( \psi \) be a step function on \([a, b] \) and \( g : [a, b] \to \mathbb{R} \) be a regulated function. Then \( \psi \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b] \).

By the Uniform Convergence Theorem and Theorem 6.2, we obtain an improvement of Theorem 5.3.

**Theorem 6.3.** Let \( f, g : [a, b] \to \mathbb{R} \) be regulated functions. If \( g \) is of bounded variation, then \( f \) is Baire One-Stieltjes integrable with respect to \( g \) on \([a, b] \).
If \( g \) is of bounded variation, then the Riemann-Stieltjes integral, \((\mathcal{B}) \int_a^b f \, dg\) defines a bounded linear functional on the space of all continuous functions \( f \) on \([a, b]\). In what follows, we prove that, for the Baire One-Stieltjes integral, this space can be extended to include all regulated functions \( f \) on \([a, b]\).

**Theorem 6.4.** Let \( f \in R([a, b]) \). If \( g : [a, b] \to \mathbb{R} \) is of bounded variation on \([a, b]\), then

\[
\left| (\mathcal{B}) \int_a^b f \, dg \right| \leq \| f \|_\infty \cdot V(g, [a, b]).
\]

**Theorem 6.5.** If \( g : [a, b] \to \mathbb{R} \) is of bounded variation, then the function \( T : R([a, b]) \to \mathbb{R} \) defined by

\[
T(f) = (\mathcal{B}) \int_a^b f \, dg
\]

is a bounded linear functional.

**References**


Received: May 15, 2014