Weakly Hull Number of a Graph

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Abstract

Given a connected graph $G$, a subset $C$ of $V(G)$ is called a weakly convex set of $G$ if for every two vertices $u, v \in C$, there exists a $u$-$v$ geodesic whose vertices belong to $C$ or equivalently, if for every two vertices $u, v \in C$, $d_{(C)}(u, v) = d_{G}(u, v)$. Let $S$ be a subset of $V(G)$. A weakly convex hull of $S$ is a weakly convex set of minimum order containing $S$.

In this paper, we introduce the concepts of weakly convex hull and weakly hull number of a graph. Moreover, we determine the weakly hull numbers of some special graphs and graphs resulting from some binary operations.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph. Let $u, v \in V(G)$. Then the distance $d_G(u, v)$ between $u$ and $v$, is the length of the shortest $u$-$v$ path $(P(u, v))$ in $G$. A $u$-$v$ path of length $d_G(u, v)$ is called $u$-$v$ geodesic. The set of all $u$-$v$ geodesics is denoted by $g_{u,v}$. The diameter $\text{diam}(G)$ of $G$ is $\max_{u,v \in V(G)} d_G(u, v)$. If $G$ is disconnected, then $\text{diam}(G) = +\infty$. A subset $C$ of $V(G)$ is called a weakly convex set of $G$ if for every two vertices $u, v \in C$, there exists $P(u, v) \in g_{u,v}$ whose elements belong to $C$ or equivalently, if for every two vertices $u, v \in C$, $d_{\langle C \rangle}(u, v) = d_G(u, v)$. This concept is introduced and studied in [4] and [5].

Let $S$ be a subset of $V(G)$. A weakly convex hull of $S$ is a weakly convex set of minimum order containing $S$. If $C$ is a convex subset of $V(G)$, then $C$ is a weakly convex set of $G$. Note that a weakly convex hull need not be unique. To see this, consider the graph $G$ in Figure 1. Let $S = \{a, c\}$. Then the sets $A_1 = \{a, b, c\}$ and $A_2 = \{a, d, c\}$ are weakly convex hulls of $S$ with $A_1 \neq A_2$.

![Figure 1: A graph $G$ with two weakly convex hulls](image)

In case where the weakly convex hull of a set $S$ is unique, we denote the weakly convex hull of $S$ by $[S]_G^w$. A set $S \subseteq V(G)$ is called a weakly hull set if $[S]_G^w = V(G)$. A weakly hull set of $G$ of minimum cardinality is a minimum weakly hull set and its cardinality is the weakly hull number of $G$ denoted by $\text{wh}(G)$.

2 Weakly Hull Number of Special Graphs

**Lemma 2.1** Let $G$ be a connected graph and $S \subseteq V(G)$. Then $S$ is weakly convex if and only if $[S]_G^w = S$.

**Lemma 2.2** Let $G$ be a connected graph. Then $\text{wh}(G) = 1$ if and only if $G = K_1$.

**Lemma 2.3** Let $G$ be a connected graph of order $n \geq 2$. Then $\text{wh}(G) = 2$ if and only if $G = P_n$. 
**Proof:** Suppose \( wh(G) = 2 \), say \( S = \{a, b\} \), is a weakly hull set of \( G \). If \( n = 2 \), then \( G = P_2 \). Suppose \( n \geq 3 \). Then, \( S \) is not weakly convex, hence \( ab \notin E(G) \). Let \( c \in V(G) \setminus S \). Then \( c \) is in some \( a-b \) geodesic \( [a_1, a_2, \ldots, a_k] \) where \( a_1 = a \), \( a_k = b \) and \( c = a_j \) for some \( j, 1 < j < k \). Since \( \{a_1, a_2, \ldots, a_k\} \) is weakly convex, by Lemma 2.1, \( [S]_G^w = \{a_1, a_2, \ldots, a_k\} \). By assumption, it follows that \( V(G) = \{a_1, a_2, \ldots, a_k\} \) and \( k = n \). Thus, \( G = P_n \).

Conversely, suppose \( G = P_n = \{a_1, a_2, \ldots, a_n\} \). Take \( S = \{a_1, a_n\} \). Then \( [S]_G^w = V(P_n) \). Thus, \( wh(G) = 2 \). \( \square \)

**Theorem 2.4** Let \( n \geq 3 \). Then

\[
wh(C_n) = \begin{cases} 4 & \text{if } n = 4 \\ 3 & \text{if } n \neq 4 \end{cases}
\]

**Proof:** Let \( C_n = [x_1, x_2, \ldots, x_n, x_1] \). By Lemma 2.2 and Lemma 2.3, \( wh(C_n) \neq 1, 2 \). We consider the following cases:

Case 1: Suppose \( n \) is odd.

Consider \( S = \{x_1, x_n, x_{n+1}\} \). Then \( [S]_G^w = V(C_n) \).

Case 2: Suppose \( n \) is even and \( n \neq 4 \).

Take \( S = \{x_1, x_{n-1}, x_{n+1}\} \). Hence, \( [S]_G^w = V(C_n) \). In either case, we have \( [S]_G^w = V(C_n) \). Therefore, \( wh(C_n) = |S| = 3 \).

Consider \( C_4 = [a, b, c, d, a] \). Note that the sets \( \{a, b, c\}, \{b, c, d\}, \{c, d, a\} \) and \( \{d, a, b\} \) are all weakly convex. By Lemma 2.1, \( \{[a, b, c]\}_C^w = \{a, b, c\} \), \( \{[b, c, d]\}_C^w = \{b, c, d\} \), \( \{[c, d, a]\}_C^w = \{c, d, a\} \) and \( \{[d, a, b]\}_C^w = \{d, a, b\} \). Hence, \( wh(C_4) \neq 3 \). Therefore, \( wh(C_4) = 4 \). \( \square \)

The following result gives a characterization of a graph \( G \) with \( wh(G) = |V(G)| \).

**Theorem 2.5** Let \( G \) be a connected graph of order \( n \). Then \( wh(G) = |V(G)| = n \) if and only if \( G = K_n \) or \( |g_{x,y}| \geq 2 \) for all \( x, y \in V(G) \) such that \( xy \notin E(G) \).

**Proof:** Let \( G \) be a connected graph of \( n \) vertices with \( wh(G) = n \). Suppose \( G \neq K_n \). Then, there exist distinct vertices \( u, v \in G \) such that \( uv \notin E(G) \). Assume \( |g_{u,v}| = 1 \). Then, there exists a unique \( u-v \) geodesic \( P_k = [a_1, a_2, \ldots, a_k] \) where \( a_1 = u, a_k = v \) and \( k \geq 3 \). It follows that there exists at least one vertex, say \( c \in P_k \) such that \( c = a_i, 1 < i < k \). Consider \( S = V(G) \setminus \{c\} \). Then, \( [S]_G^w = V(G) \) and

\[
wh(G) \leq |S| = |V(G)| - 1 = n - 1 \neq n,
\]

a contradiction to the above assumption. Thus, \( |g_{x,y}| \geq 2 \) for all \( x, y \in V(G) \) such that \( xy \notin E(G) \).
Conversely, let $G = K_n$. Then for every subset of $V(G)$, its weakly hull set is itself. Hence, $V(G)$ is the only weakly hull set in $G$ and $wh(G) = n$. Suppose $G \neq K_n$. Let $S \subseteq V(G)$. Then there exists $c \in V(G) \setminus S$. Take $S^* = V(G) \setminus \{c\}$. Then $S \subset S^*$. Let $a, b \in S^*, a \neq b$. If there exists no $a$-$b$ geodesic containing $c$, then every vertex of an $a$-$b$ geodesic belongs to $S^*$. Suppose there exists an $a$-$b$ geodesic $[x_1, x_2, \ldots, x_k]$ such that $x_1 = a$, $x_k = b$ and $c = x_i, 1 < i < k$. By assumption, there exists $[x_{i-1}, y, x_{i+1}] \in g_{x_{i-1}, x_{i+1}}$ such that $y \neq c$. Hence, $[x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k]$ is an $a$-$b$ geodesic whose vertices are contained in $S^*$. Therefore, $S^*$ is a weakly convex set. Since $S \subseteq S^*$, it follows that $[S]_G^w \subseteq S^*$. Consequently, $V(G)$ is the only weakly hull set in $G$. Therefore, $wh(G) = n$. 

**Lemma 2.6** If $S$ is a weakly hull set of $G$, then $Ext(G) \subseteq S$.

**Proof:** Let $S$ be a weakly hull set of $G$. Suppose there exists $a \in Ext(G) \setminus S$. Since $a$ cannot be in any $x$-$y$ geodesic, where $x \neq a$ and $y \neq a$, it follows that $a \notin [S]_G^w$. This implies that $[S]_G^w \neq V(G)$, a contradiction to our assumption. Therefore, $Ext(G) \subseteq S$. 

The next corollaries follow from Lemma 2.6 since every vertex of a complete graph is extreme and every leaf is an extreme vertex.

**Corollary 2.7** If $n \geq 1$, then $wh(K_n) = n$.

**Corollary 2.8** If $S$ is a weakly hull set of $G$, then $L(G) \subseteq S$ where $L(G)$ is the set of leaves of $G$.

**Theorem 2.9** Let $T_n$ be a tree of order $n \geq 1$. Then $wh(T_n) = |L(T_n)|$.

**Proof:** Clearly, $L(T_n)$ is a weakly hull set of $T_n$. By Corollary 2.8, it follows that $L(T_n)$ is the minimum hull set of $T_n$. Therefore, $wh(T_n) = |L(T_n)|$. 

3 A Realization Problem

Let $G$ be a connected graph and $S$ be a subset of $V(G)$. The convex hull of $S$, denoted by $[S]_G$ is the smallest convex set in $G$ containing $S$. A set $S \subseteq V(G)$ is called a hull set if $[S]_G = V(G)$. A hull set of $G$ of minimum cardinality is a minimum hull set and its cardinality is the hull number denoted by $h(G)$. These concepts are investigated in [1], [2], and [3].

**Theorem 3.1** [3] If $G$ is a connected graph with $n$ vertices, then $h(G) = n$ if and only if $G$ is complete.
Theorem 3.2 Let $G$ be a connected graph. Then $h(G) \leq wh(G)$.

Proof: Let $S \subseteq V(G)$. The convex hull $[S]_G$ of $S$ is the smallest convex set containing $S$. It follows that $[S]_G$ is a weakly convex set containing $S$. Consequently, if $S_1$ is a weakly convex hull of $S$, then $S_1 \subseteq [S]_G$. Thus, in particular, if $S$ is a weakly hull set of $G$, then $V(G) = [S]_G^{wh} \subseteq [S]_G$. This implies that $S$ is a hull set of $G$. Therefore, $h(G) \leq wh(G)$. \qed

The next result shows that every pair of positive integers are realizable as values of the parameters hull number and weakly hull number.

Theorem 3.3 Let $a$ and $b$ be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph $G$ with $h(G) = a$ and $wh(G) = b$.

Proof: Consider the following cases:
Case 1: Suppose $a = b$.
Consider $G_1 = K_a$. By Theorem 3.1 and Corollary 2.7, it follows that $h(G_1) = wh(G_1) = a$.
Case 2: Suppose $2 = a < b$.
Let $G_2$ be a graph obtained from the path $[x, z, y]$ by adding paths $[x, v_i, z]$ for $1 \leq i \leq b - a$ (see Figure 2).

![Figure 2: The graph $G_2$](image)

The sets $S_1 = \{x, y\}$ and $S_2 = S_1 \cup \{v_i : i = 1, 2, \ldots, b-a\}$ are, respectively, the minimum hull and minimum weakly hull sets of $G_2$. Thus, $h(G_2) = |S_1| = 2 = a$ and $wh(G_2) = a + (b-a) = b$.
Case 3: Suppose $2 < a < b$.
Consider the graph $G_3$ obtained from $G_2$ in Figure 2 by adding the edges $x_i x$ for $i = 1, 2, \ldots, a - 1$ (see Figure 3).
Let $S_3 = \{x_i : i = 1, 2, \ldots, a-1\} \cup \{y\}$ and $S_4 = S_3 \cup \{v_j : j = 1, 2, \ldots, b-a\}$. Then $S_3$ is a minimum hull set and $S_4$ is a minimum weakly hull set of $G$. Therefore $h(G) = |S_3| = a$ and $wh(G) = a + (b - a) = b$. □

The following corollaries follows directly from Theorem 3.3.

**Corollary 3.4** Given a positive integer $n$, there exists a connected graph $G$ such that $wh(G) - h(G) = n$, that is, the difference $wh - h$ can be made arbitrarily large.

**Corollary 3.5** For every pair of positive integers $a$ and $b$ with $2 \leq a < b$, the smallest possible order of a connected graph $G^*$ such that $wh(G^*) = a$ and $h(G^*) = b$ is $b + 2$.

### 4 Weakly Hull Number of the Join of Graphs

Let $A$ and $B$ be sets which are not necessarily disjoint. The disjoint union of $A$ and $B$, denoted by $A \cup B$, is the set obtained by taking the union of $A$ and $B$ treating each element in $A$ as distinct from each element in $B$. The join of two graphs $G$ and $H$ is the graph $G + H$ with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

**Theorem 4.1** Let $G$ be a nontrivial graph of order $n$. Then

$$wh(K_1 + G) = \begin{cases} n + 1 & \text{if } diam(G) \leq 2 \\ n & \text{if otherwise} \end{cases}$$

**Proof:** Let $v \in V(K_1)$. Suppose $diam(G) \leq 2$. If $diam(G) = 1$, then $G$ is complete. Hence, by Theorem 2.5, $wh(K_1 + G) = |V(K_1 + G)| = n + 1$. Assume $diam(G) = 2$. Then there exist $x, y \in V(G)$ such that $d_G(x, y) = 2$, that
is, \([x, z, y] \in g_{x,y}\) for some \(z \in V(G)\). By definition of a join, it follows that \([x, v, y] \in g_{x,y}\). Thus, \(|g_{x,y}| \geq 2\). By Theorem 2.5, \(wh(K_1 + G) = n + 1\). Suppose \(diam(G) \geq 3\). Let \(S\) be a minimum weakly hull set of \(K_1 + G\). Suppose \(v \in S\). Suppose further that \(V(G) \setminus S \neq \emptyset\). Then, since \(S = \{v\} \cup (V(G) \cap S)\) is weakly convex, it follows that a weakly convex hull of \(S\) is \(S\), contrary to our assumption. Thus, \(V(G) \cap S = V(G)\) and \(S = V(K_1 + G)\). This, however, is not possible because \(V(G)\) is also a weakly hull set of \(K_1 + G\). Thus, \(v \notin S\). This implies that \(S \subseteq V(G)\). Assume \(S \setminus V(G) \neq \emptyset\). Since \(\{v\} \cup S\) is a weakly convex hull in \(K_1 + G\), it follows that \([S]_G^w = S \cup \{v\} \neq V(G)\). This is a contradiction. Therefore, \(S = V(G)\) and \(wh(K_1 + G) = n\). □

The following corollary is a direct consequence of the above theorem.

**Corollary 4.2** Let \(n\) be a positive integer.

(i) \[wh(F_n) = \begin{cases} n + 1 & \text{if } 1 \leq n \leq 3 \\ n & \text{if } n \geq 4 \end{cases}\]

(ii) \[wh(W_n) = \begin{cases} n + 1 & \text{if } 3 \leq n \leq 5 \\ n & \text{if } n \geq 6 \end{cases}\]

(iii) If \(n \geq 2\), then \(wh(K_{1,n}) = n\).

**Theorem 4.3** Let \(G\) and \(H\) be nontrivial graphs of orders \(m \geq 2\) and \(n \geq 2\), respectively. Then, \(wh(G + H) = m + n\).

**Proof:** If \(G + H\) is a complete graph, then by Theorem 2.5, \(wh(G + H) = |V(G + H)| = m + n\).

Suppose \(G + H \neq K_{m+n}\). Then there exist \(x, y \in V(G + H)\) such that \(xy \notin E(G + H)\). Since \(x\) and \(y\) are not adjacent in \(G + H\), it follows that \(x, y\) belong to the same vertex set, say \(V(G)\) or \(V(H)\). Otherwise, if \(x \in V(G)\) and \(y \in V(H)\), then \(xy \in E(G + H)\), a contradiction. Suppose \(x, y \in V(G)\). Since \(H\) is nontrivial, there exist \(w, z \in V(H)\) such that \([x, w, y], [x, z, y] \in g_{x,y}\). Thus, \(|g_{x,y}| \geq 2\). By Theorem 2.5, \(wh(G + H) = m + n\). Similarly, if \(x, y \in V(H)\), it follows that \(wh(G + H) = m + n\). □

The following corollary follows directly from Theorem 4.3.

**Corollary 4.4** Let \(m\) and \(n\) be positive integers greater than 1. Then \(wh(K_{m,n}) = m + n\).
References


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