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Hyers-Ulam Stability of Abstract Second Order Linear Dynamic Equations on Time Scales

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Abstract

In this paper we investigate the Hyers-Ulam Stability of the abstract dynamic equation of the form

$$x^{\Delta\Delta}(t) + A(t)x^{\Delta}(t) + R(t)x(t) = f(t), \quad t \in \mathbb{T},$$

where $A, R: \mathbb{T} \to \mathrm{L}(\mathbb{X})$, the space of all bounded linear operators from a Banach space $\mathbb{X}$ into itself, and f is rd-continuous from a time scale $\mathbb{T}$ to $\mathbb{X}$. Some examples illustrate the applicability of the main result.

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1. INTRODUCTION

Ulam [16] posed the following problem concerning the stability of functional equations: under what conditions does a linear mapping near an approximately linear mapping exist? The problem for the case of approximately mappings was solved by Hyers [10],who proved that Cauchy equation is stable in Banach spaces, and the results were generalized by Rassias [14].

Alsina and Ger [8] were the first authors who investigated the Hyers-Ulam stability of a differential equation. Many articles dealing with Ulam, Hyers-Ulam, and Hyers-Ulam- Rassias stability in various contexts were edited by Rassias [15]. In [11] Li and Shen introduced the Hyers-Ulam stability of scalar second order differential equations of the form

$$y'' + p(x)y' + q(x)y + r(x) = 0,$$

that is, if f is an approximated solution of the last equation, then there exists an exact solution of the equation near f.

Also Li and Shen [12] proved the Hyers-Ulam stability of homogeneous linear differential equations of second order.

In [7] Pasc Gavruta proved the Hyers-Ulam stability of second order linear differential equation with boundary conditions or initial conditions. That is if y is an approximate solution of the differential equation

$$y'' + \beta(x)y = 0 \qquad with\ \ y(a) = y(b),$$

then there exists a solution of the differential equation near y.

In [9] Mohamed Bagher proved the Hyers-Ulam stability of exact second –order linear differential equations. Also he showed the Hyers-Ulam stability of the following equations**:**

Second–order linear differential equations with constant coefficients, Euler, Hermite's, Cheybyshev and Legendre's differential equations. These results generalize the main results of Li and Shen [11-12].

In [13] Maher established the Hyers-Ulam stability of generalized nonlinear second-order differential equations and he proved the Hyers-Ulam stability of Emden–Fowler type equation with initial conditions.

In 2012 Douglas and Ben Gates [4] established the Hyers-Ulam stability of the scalar second-order linear non-homogeneous dynamic equation on time scales of the form:

$$x^{\Delta\Delta}(t) + p(t)x^{\Delta}(t) + r(t)x(t) = f(t), t \in [a,b]_{\mathbb{T}},$$

where the given functions p, r, $f \in C_{rd}\,[a,b]_{\mathbb{T}}$, the space of all real valued rd-continuous on $[a,b]_{\mathbb{T}}$. Here $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$. They extended the work of Li and Shen [11,12] to prove the Hyers-Ulam stability of homogenous and non-homogenous linear dynamic equations of second order on time scales.

In this paper we generalize and extend the work of Douglas R. Anderson, Ben Gates and Dylan Heuer [4] and we investigate the Hyres-Ulam stability of the abstract dynamic equation of the form:

$$x^{\Delta\Delta}(t) + A(t)x^{\Delta}(t) + R(t)x(t) = f(t), \quad t \in \mathbb{T},$$

where $A, R \in \mathrm{C_{rd}}\big(\mathbb{T}, \mathrm{L}(\mathbb{X})\big)$, the space of all rd-continuous from a time scale $\mathbb{T}$ to $\mathrm{L}(\mathbb{X})$, and $f \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{X})$. Here $\mathrm{L}(\mathbb{X})$ is the space of all bounded linear operators from a Banch space

$\mathbb{X}$ into itself.

2. Preliminaries

We need the following definitions and notations from [5] in proving our main results in Section 3.

Definition2.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Definition 2.2. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = inf\{s \in \mathbb{T}: s > t\}$, and $\rho(t) = sup\{s \in \mathbb{T}: s < t\}$ are called the jump operators.

Definition2.3. A point $t \in \mathbb{T}$ is said to be right–dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense *if* $\rho(t) = t$, left-scattered *if* $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$, and dense if $\rho(t) = t = \sigma(t)$.

Definition2.4. Let $t \in \mathbb{T}$. The graininess function

$\mu: \mathbb{T} \to [0, \infty[$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 2.5. A function $f: \mathbb{T} \to \mathbb{X}$ is called rd-continuous provided

(i) f is continuous at every right-dense point;

(ii) $\lim_{s\to t^-} f(s)$ exists (finite) at left-dense points in $\mathbb{T}$.

The set of rd-continuous functions $f: \mathbb{T} \to \mathbb{X}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{X})$.

Definition 2.6. (The Delta Derivative). A function $f: \mathbb{T} \to \mathbb{X}$ is called Δ-differentiable at $t \in \mathbb{T}^k$if there exists an element $f^{\Delta}(t) \in \mathbb{X}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that:

$$\left\|\left[f\big(\sigma(t)\big) - f(s)\right] - f^{\Delta}(t)[\sigma(t) - s]\right\| \leq \varepsilon|\sigma(t) - s|, s \in (t - \delta, t + \delta) \cap \mathbb{T}.$$

In this case $f^{\Delta}(t)$is called the delta derivative of *f* at *t*, provided it exists and we have

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f\big(\sigma(t)\big) - f(s)}{\sigma(t) - s}. \qquad (2.1)$$

If$f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$, we say that *f* is delta differentiable on$\mathbb{T}^k$. Here

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}] & if \sup \mathbb{T} < \infty \\ \mathbb{T} & if \sup \mathbb{T} = \infty. \end{cases}$$

We also denote by

$$C_{rd}^n(\mathbb{T}, \mathbb{X}) = \left\{x \in C_{rd}(\mathbb{T}, \mathbb{X}) \colon x^{\Delta}, \ldots, x^{\Delta^n} \text{exist and belong to } C_{rd}(\mathbb{T}, \mathbb{X})\right\}.$$

Definition2.7. A function $f: \mathbb{T} \to \mathbb{X}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in$\mathbb{T}$.

Definition 2.8. Let $f: \mathbb{T} \to \mathbb{X}$ be a regulated function. Any function *F* which satisfies $F^{\Delta}(t) = f(t)$for all $t \in \mathbb{T}^k$, is called a pre-antiderivative of f. The indefinite integral of a regulated function *f* is defined by

$$\int f(t)\Delta t = F(t) + C, \qquad (2.2)$$

where *C* is an arbitrary constant. The Cauchy integral of f is defined by

$$\int_r^s f(t)\,\Delta t = F(s) - F(r), \quad r, s \in \mathbb{T}. \tag{2.3}$$

Definition 2.9. A function $p: \mathbb{T} \to \mathbb{R}$ is said to be regressive provided

$$1 + \mu(t)p(t) \neq 0\,, \qquad \textit{for all } t \in \mathbb{T}.$$

The set of all regressive functions $f: \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.10(The Generalized Exponential Function).

If $p \in \mathcal{R}$, then the exponential function $e_p(t,s)$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(t)}\left(p(\tau)\right)\Delta\tau\right), \qquad for\ s, t \in \mathbb{T},$$

where

$$\xi_{\mu(s)}\big(p(s)\big) = \begin{cases} \dfrac{1}{\mu(s)} log(|1 + \mu(s)p(s)| + i\ Arg(1 + \mu(s)p(s))), & for\ \mu(s) > 0 \\ p(s) & \mu(s) = 0 \end{cases}$$

Definition 2.11. Let $F: \mathbb{T} \times \mathbb{X}^n \to \mathbb{X}$. The n^{th}order dynamic equation

$$x^{\Delta^n}(t) = F\big(t, x^{\Delta^{n-1}}, \ldots x^{\Delta}, x\big), \;\; t \in \mathbb{T}. \tag{2.4}$$

is said to have Hyers-Ulam stability on $\mathbb{T}$ if for every $\varepsilon > 0$ and $u \in C_{rd}^n(\mathbb{T}, \mathbb{X})$ which satisfies

$$\left\| u^{\Delta^n}(t) - F(t, u^{\Delta^{n-1}}(t) \ldots, u^{\Delta}(t), u) \right\| < \varepsilon, \qquad t \in \mathbb{T},$$

there exists a solution x *of* (2.4) such that:

$$\|u(t) - x(t)\| < L\varepsilon\,, \qquad t \in \mathbb{T} \text{ for some } L > 0.$$

We need the following theorem in proving our main results.

Theorem2.1. *[2] Assume* that $A \in C_{rd}\big(\mathbb{T}, \mathrm{L}(\mathbb{X})\big), f \in C_{rd}(\mathbb{T}, \mathbb{X})$. *In addition we assume that A satisfies the following conditions:*

(1) $sup_t \|A(t)\| < \infty$

(2) A is regressive, that is, $(I+\mu(t)A(t))$ *is invertible for every* $t\in\mathbb{T}$*, where* I *is the identity operator.*

If the function

$$F(t)=\int_{t_0}^{t}\left\|e_A\big(t,\sigma(s)\big)\right\|\Delta s, t_0\in\mathbb{T} \tag{2.5}$$

is bounded, then the equation

$$x^{\Delta}(t)=A(t)x(t)+f(t),\ \ t\in\mathbb{T}\,,\qquad t>t_{0,}$$

$$x(t_0)=x_0\in\mathbb{X} \tag{2.6}$$

has Hyers-Ulam stability, That is, for every $\varepsilon>0,\ g\in C_{rd}^{1}(\mathbb{T},\mathbb{X})$ *which satisfies:*

$\left\|g^{\Delta}(t)-A(t)g(t)-f(t)\right\|<\varepsilon, t\in\mathbb{T},\ t\geq t_{0,}$

there exists a solution $\boldsymbol{v}\in\boldsymbol{C}_{\boldsymbol{rd}}^{\Delta}(\mathbb{T},\mathbb{X})$ ***of*** *Equation* $(\mathbf{2.6})$ *such that*

$\|\boldsymbol{g}(\boldsymbol{t})-\boldsymbol{v}(\boldsymbol{t})\|<\boldsymbol{L\varepsilon},\ \boldsymbol{t}\in\mathbb{T}\,,\boldsymbol{t}\geq\boldsymbol{t_0}$ *for some constant* $\boldsymbol{L}>\mathbf{0}$.
The properties of the operator exponential function $e_A(t,s)$ were established in [1] and [3].

3. HYERS-ULAM STABILITY

In this section we establish the Hyers –Ulam stability of second order non-homogeneous dynamic equations of the form

$$x^{\Delta\Delta}(t)+A(t)x^{\Delta}(t)+R(t)x(t)=f(t),\quad t\in\mathbb{T}, \tag{3.1}$$

where $A\,,R:\mathbb{T}\to L(\mathbb{X})$ and $f\in C_{rd}(\mathbb{T},\mathbb{X})$.

Theorem3.1. *Assume there is a particular solution* $z:\mathbb{T}\to L(\mathbb{X})$ *of the corresponding Recatti equation*

$$z^{\Delta}(t)+\big(A(t)-z^{\sigma}(t)\big)z(t)=R(t)\,,\qquad t\in\mathbb{T}\,, \tag{3.2}$$

such that $D=z^{\sigma}-A,\ -z$ *are regressive and* $D\in C_{rd}(\mathbb{T},\mathbb{X})$.

If the following functions:

$$F_1(t) = \int_{t_0}^{t} \|e_D(t, \sigma(s))\| \Delta s, t_0 \in \mathbb{T} \quad (3.3)$$

and

$$F_2(t) = \int_{t_0}^{t} \|e_{-z}(t, \sigma(s))\| \Delta s, t_0 \in \mathbb{T} \quad (3.4)$$

are bounded, then Equation (3.1) has Hyers-Ulam stability.

Proof : To show that Equation (3.1) has Hyers–Ulam stability, let $\varepsilon > 0$ and $y \in C_{rd}^2(\mathbb{T}, \mathbb{X})$ that satisfies

$$\left\| y^{\Delta\Delta}(t) + A(t)y^{\Delta}(t) + R(t)y(t) - f(t) \right\| < \varepsilon\,, t \in \mathbb{T}. \quad (3.5)$$

Our aim is to find a solution $u: \mathbb{T} \to \mathbb{X}$ *of* Equation (3.1) *such that* $\|y - u\| < k\varepsilon$ on$\mathbb{T}$ for some constant $k > 0$.

Assume that *z* is a particular solution of Equation (3.2) that satisfies the hypothesis of the Theorem.

$Set\ g = y^{\Delta} + zy$.Then $g^{\Delta} = y^{\Delta\Delta} + z^{\sigma}y^{\Delta} + z^{\Delta}y$. Consequently, we have

$$\left\| g^{\Delta} - Dg - f \right\| = \left\| y^{\Delta\Delta} + z^{\sigma}y^{\Delta} + z^{\Delta}y(t) - (z^{\sigma} - A)(y^{\Delta} + zy) - f \right\|$$

$$= \left\| y^{\Delta\Delta} + Ay^{\Delta} + Ry - f \right\| < \varepsilon \quad \text{on } \mathbb{T} \text{ (from (3.5))}.$$

Since F_1 is bounded, then by Theorem 2.1 the equation

$$x^{\Delta}(t) - D(t)x(t) = f(t), \quad t \in \mathbb{T} \quad (3.6)$$

$$x(t_0) = x_0$$

has Hyers-Ulam stability, and there is a solution $\mathrm{v} \in \mathrm{C}_{\mathrm{rd}}^1(\mathbb{T}, \mathbb{X})$of Equation (3.6) such that

$\|g(t) - v(t)\| < L\varepsilon, \quad t \in \mathbb{T}\ \ , t \geq t_0$ for some $L > 0$. It follows that

$\left\|y^{\Delta}(t)+z(t)y(t)-v(t)\right\|<L\varepsilon, \qquad t\geq t_0\,.$

Again, in view of the boundedness of F_2, there is a solution $u\ \in C_{rd}^1(\mathbb{T},\mathbb{X})$of the equation

$$x^{\Delta}(t)+z(t)x(t)-v(t)=0, \quad t>t_0\,, \tag{3.7}$$

$$x(t_0)=\ x_0$$

such that $\|y(t)-u(t)\|<kL\varepsilon,\ t\in\mathbb{T}, t>t_0$ for some $k\ >0$. Now, we can cheek that u is a solution of Equation (3.1). Indeed, we have

$$u^{\Delta\Delta}+Au^{\Delta}+Ru-f=\ (v^{\Delta}-z^{\sigma}u^{\Delta}-z^{\Delta}u-Au^{\Delta}+Ru-f)$$

$$=(Dv+f)-(D+A)u^{\Delta}+(R-z^{\Delta})u+Au^{\Delta}-f$$

$$=D(v-u^{\Delta}-zu)=0.$$

Therefore, u is a solution of (3.1).

As a direct consequence, in view of the boundedness of $F_1(t)=$ $\int_{t_0}^{t}\|e_D(t,\sigma(s))\|\Delta s\ and\ F_2(t)=\int_{t_0}^{t}\|e_{-z}(t,\sigma(s))\|\Delta s$ on$[a,b]_{\mathbb{T}}$,

we obtain the following result

Corollary3.2. The equation

$$x^{\Delta\Delta}(t)+A(t)x^{\Delta}(t)+R(t)x(t)=f(t),\ \ t\in[a,b]_{\mathbb{T}},\ \ t>t_0 \tag{3.8}$$

has Hyers-Ulam stability.

The previous result yields the result in [4], when both of f, A and R are scalar functions.

4. Illustrative examples

The following examples show the applicability of the main result.

Example4.1 Consider the following dynamic equation

$$x^{\Delta\Delta}(t) + A(t)x^{\Delta}(t) + R(t)x(t) = 0, \quad t \in \mathbb{T} = \mathbb{R}^{\geq 0} \tag{4.1}$$

where $A(t)$ and $R(t)$ are the matrices defined by

$$A(t) = \begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix}, \ R(t) = \begin{bmatrix} 1+t^2 & 0 \\ 0 & 1+t^2 \end{bmatrix}.$$

The corresponding Ricatti Equation is

$$z^{\Delta} - (z^{\sigma}(\mathrm{t}) - A(t))z(t) = \mathrm{R(t)}, \quad t \in \mathbb{T}. \tag{4.2}$$

One can see that $z(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$ is a solution of Equation (4.2). We have,

$$D(t) = z^{\sigma}(\mathrm{t}) - A(t) = \begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix} = -z(t), t \in \mathbb{T}.$$

We use the formula in [1, 3, 6], for $\mathbb{T} = \mathbb{R}^{\geq 0}$, to get $e_D\big(t, \sigma(s)\big) = e_D(t,s) = e^{\frac{-(t^2-s^2)}{2}} I = e_{-z}(t,s)$. This implies that the functions F_1 and F_2, defined by (3.3) and (3.4) respectively, are bounded. Therefore, by Theorem 3.1, Equation (4.1) has Hyres-Ulam stability on $\mathbb{T} = \mathbb{R}^{\geq 0}$.

Example 4.2 Consider the following dynamic equation

$$x^{\Delta\Delta}(t) + A(t)x^{\Delta}(t) = 0, \quad t \in \mathbb{T} = [a, \infty[\,, a > 0, \tag{4.3}$$

$$x(t_0) = x_0, \ \ x^{\Delta}(t_0) = x_1$$

where $A(t)$ is the 2x2 matrix defined by

$$A(t) = \begin{bmatrix} -t - \frac{1}{t} & 0 \\ 0 & 0 \end{bmatrix}$$

The corresponding Ricatti equation is:

$$z^{\Delta} - (z^{\sigma}(\mathrm{t}) - A(t))z(t) = 0, \quad t \in \mathbb{T} \tag{4.4}$$

Clearly $z(t) = \begin{bmatrix} -t & 0 \\ 0 & 0 \end{bmatrix}$ is a solution of Equation (4.4) and $z^{\sigma}(\mathrm{t}) = \begin{bmatrix} -t & 0 \\ 0 & 0 \end{bmatrix}. Then$

$D = (z^{\sigma}(\text{t}) - A(t)) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & 0 \end{bmatrix}$. The matrix exponential function $e_D\big(t, \sigma(s)\big) = e_D(t,s)$ is given by

$$e_D(t,s) = \begin{bmatrix} \frac{t}{s} & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\|e_D(t,s)\| = \sqrt{(\frac{t}{s})^2 + 1}$ which implies that, $F_1(t) = \int_a^t \sqrt{(\frac{t}{s})^2 + 1}\, ds$ is unbounded. One can see that Equation (4.3) does not have Hyers-Ulam stability. Indeed,

let $\varepsilon > 0$ and $g = \begin{pmatrix} \varepsilon \\ \frac{\varepsilon t}{6} \end{pmatrix}$. We have $\left\|g^{\Delta\Delta} + Ag^{\Delta}\right\| = \left\|\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\| < \varepsilon$. To get a solution of Equation (4.3), take a change of variable $x^{\Delta} = v$. We obtain the equation

$v^{\Delta} = -Av$, $v(t_0) = v_{0,}$ whose solution is $v(t) = e_{-A}(t, t_0)v_0$. For $t_0 = 1$, $v_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have $e_{-A}(t, t_0) = e_{-A}(\text{t}, 1) = \begin{bmatrix} e^{[\frac{\text{t}^2}{2} + \ln t - \frac{1}{2}]} & 0 \\ 0 & 1 \end{bmatrix}$ and $v(t) = \begin{pmatrix} e^{[\frac{\text{t}^2}{2} + \ln t - \frac{1}{2}]} \\ 1 \end{pmatrix} = \begin{pmatrix} te^{\frac{\text{t}^2}{2} - \frac{1}{2}} \\ 1 \end{pmatrix}$. The Equation

$x^{\Delta} = \begin{pmatrix} te^{\frac{\text{t}^2}{2} - \frac{1}{2}} \\ 1 \end{pmatrix}$, $x(t_0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, has the solution given by $x(t) = \begin{pmatrix} e^{\frac{\text{t}^2}{2} - \frac{1}{2}} \\ t \end{pmatrix}$.

We have

$\|g(t) - x(t)\| = \sqrt{(\varepsilon - e^{\frac{\text{t}^2}{2} - \frac{1}{2}})^2 + (\frac{\varepsilon t}{6} - t)^2} \to \infty \; as \; t \to \infty.$

Therefore, Equation (*4.3*) does not have Hyers- Ulam stability.

Example4.3 Consider the scalar dynamic equation

$$x^{\Delta\Delta}(t) + 5x^{\Delta}(t) + 6x(t) = 6, \quad t \in \mathbb{T}, \qquad (4.5)$$

Here, $A(t) = 5$, $R(t) = 6$ and $f(t) = 6$.

Case1: $\mathbb{T} = \mathbb{R}^{\geq 1}$.

The corresponding Recatti equation :

$$z^{''}(t) + 5z(t) - z^2(t) = 6$$

has a particular solution $z(t): \mathbb{T} \to \mathbb{R}$ which is given by $z = \frac{1}{1-e^t} + 2$. Now we check the boundedness of F_1, F_2 defined by (3.3) and (3.4). We have $D(t) = z^\sigma(t) - A(t) = \frac{1}{1-e^t} - 3$, $e_D(t,s) = \frac{e^{-s}-1}{e^{-t}-1} e^{-3(t-s)}$ and , $e_{-Z}(t,s) = \frac{1-e^{-t}}{1-e^{-s}} e^{-2(t-s)}$. We can see easily that $F_1(t) = \int_1^t \frac{1-e^{-s}}{1-e^{-t}} e^{-3(t-s)} \Delta s$ and $F_2(t) = \int_1^t \frac{1-e^{-t}}{1-e^{-s}} e^{-2(t-s)} \Delta s$ are bounded functions. Therefore, Equation (4.5) has Hyers-Ulam stability.

Case 2: $\mathbb{T} = \mathbb{Z}^{\geq 0}$.

Now we can check that Equation (4.5) does not have Hyers-Ulam stability**.** Let $\varepsilon > 0$ and $g(t) = \frac{\varepsilon}{30 \times 2^{t+1}}$. This implies that $\left|g^{\Delta\Delta} + 5g^\Delta + 6g\right| = \left|\frac{\varepsilon}{3 \times 2^{t+4}}\right| < \varepsilon,\ t \in \mathbb{T} = \mathbb{Z}^{\geq 0}$.

One can see that $u(t) = 3(-2)^t - 2(-1)^t + 1$ is a solution of Equation (4.5).We have $|g(t) - u(t)| = \left|\frac{\varepsilon}{30 \times 2^{t+1}} - 3(-2)^t + 2(-1)^t - 1\right|$. When t is odd, $|g(t) - u(t)| = \left|\frac{\varepsilon}{30 \times 2^{t+1}} + 3(2)^t - 3\right| \to \infty$ $as\ t \to \infty$, and when t is even, $|g(t) - u(t)| = \left|\frac{\varepsilon}{30 \times 2^{t+1}} - 3(2)^t + 1\right| \to \infty\ as\ t \to \infty$. Consequently, $|g(t) - u(t)| \to \infty\ as\ t \to \infty$.

Therefore, Equation (4.5) does not have hyers –Ulam stability.

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