On Annulus Containing all the Zeros of a Polynomial

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Abstract

Recently Dalal and Govil [5] proved that, for any set of positive numbers \( \{A_k\}_{k=1}^{n} \) such that \( \sum_{k=1}^{n} A_k = 1 \), a complex polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) \( (d_k \neq 0, \ 0 \leq k \leq n) \) has all its zeros in the annulus \( A = \{ z : r_1 \leq |z| \leq r_2 \} \), where

\[
r_1 = \min_{1 \leq k \leq n} \left\{ A_k \left\| \frac{d_0}{d_k} \right\| \right\}^{\frac{1}{k}} \quad \text{and} \quad r_2 = \max_{1 \leq k \leq n} \left\{ \frac{1}{A_k} \left\| \frac{d_n-k}{d_n} \right\| \right\}^{\frac{1}{k}}.
\]

This paper presents the best possible results in the same direction by identifying the smallest annulus containing all the zeros of \( P(z) \).

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1 Introduction

Over the decades the location of zeros of a polynomial has been an important area of research for engineers as well as mathematicians, and numerous results have been reported in the literature. To mention only the results relevant to the study of this paper, Diaz-Barrero [1] proved that all the zeros of a complex
monic polynomial \( P(z) = z^n + \sum_{k=0}^{n-1} d_k z^k \) \((d_k \neq 0, \ 0 \leq k \leq n - 1)\) lie in the disks \( C_1 = \{ z : |z| \leq r_1 \} \) or \( C_2 = \{ z : |z| \leq r_2 \} \), where

\[
r_1 = \max_{1 \leq k \leq n} \left\{ \frac{2^{n-1} C(n+1, 2)}{k^2 C(n, k)} |d_{n-k}| \right\}^{\frac{1}{2}} 
\]

and

\[
r_2 = \max_{1 \leq k \leq n} \left\{ \frac{F_{3n}}{C(n, k) 2^k F_k} |d_{n-k}| \right\}^{\frac{1}{2}}. 
\]

Here \( C(n, k) = \frac{n!}{k!(n-k)!} \) are the binomial coefficients, and \( F_k \) is the \( k \)th Fibonacci number defined by \( F_0 = 0, \ F_1 = 1 \) and \( F_k = F_{k-1} + F_{k-2} \) for \( k \geq 2 \).

In deriving (1) and (2), the following two identities were respectively used:

\[
\sum_{k=1}^{n} k^2 C(n, k) = 2^{n-1} C(n+1, 2) 
\]

(3)

\[
\sum_{k=1}^{n} C(n, k) 2^k F_k = F_{3n}. 
\]

(4)

Diaz-Barrero [2] also showed that a complex polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) \((d_k \neq 0, \ 0 \leq k \leq n)\) has all its zeros in the annulus \( C = \{ z : r_1 \leq |z| \leq r_2 \} \), where

\[
r_1 = \frac{3}{2} \min_{1 \leq k \leq n} \left\{ \frac{2^n F_k C(n, k) |d_0|}{F_{4n} |d_k|} \right\}^{\frac{1}{2}} 
\]

and

\[
r_2 = \frac{2}{3} \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}}{2^n F_k C(n, k)} |d_{n-k}| \right\}^{\frac{1}{2}}. 
\]

(5) and (6) were obtained based on the identity

\[
\sum_{k=1}^{n} 2^{n-k} 3^k F_k C(n, k) = F_{4n}. 
\]

(7)

A result containing the bounds (2), (5) and (6) as special cases was given by Diaz-Barrero and Egozcue [3].

On the other hand, based on the identity

\[
\sum_{k=1}^{n} C(n, k) = 2^n - 1 
\]

(8)

Kim [4] proved that all the zeros of a complex polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) \((d_k \neq 0, \ 0 \leq k \leq n)\) are contained in the annulus \( A = \{ z : r_1 \leq |z| \leq r_2 \} \) where

\[
r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C(n, k) |d_0|}{2^{n-1} |d_k|} \right\}^{\frac{1}{k}} 
\]

(9)
and

\[ r_2 = \max_{1 \leq k \leq n} \left\{ \frac{2^{n-1}}{C(n,k)} \left| \frac{d_{n-k}}{d_n} \right| \right\}^{\frac{1}{k}}. \]  

(10)

Recently Dalal and Govil [5] presented a unified result which contains all the above-mentioned approaches as special cases. More precisely they proved the following theorem.

**Theorem 1.1** Let \( \{A_k\}_{k=1}^n \) be an arbitrary set of positive numbers such that \( \sum_{k=1}^n A_k = 1 \). Then a complex polynomial \( P(z) = \sum_{k=0}^n d_k z^k \) \((d_k \neq 0, \ 0 \leq k \leq n)\) has all its zeros in the annulus \( A = \{ z : r_1 \leq |z| \leq r_2 \} \), where

\[ r_1 = \min_{1 \leq k \leq n} \left\{ \frac{A_k}{d_0} \left| \frac{d_0}{d_k} \right| \right\}^{\frac{1}{k}} \]  

(11)

and

\[ r_2 = \max_{1 \leq k \leq n} \left\{ \frac{1}{A_k} \left| \frac{d_{n-k}}{d_n} \right| \right\}^{\frac{1}{k}}. \]  

(12)

As applications of Theorem 1.1, Dalal and Govil [5] obtained the following two corollaries.

**Corollary 1.1** Let \( L_k \) be the \( k \)th Lucas number defined by \( L_0 = 2, L_1 = 1, \) and \( L_k = L_{k-1} + L_{k-2} \) for \( k \geq 2 \). Then a complex polynomial \( P(z) = \sum_{k=0}^n d_k z^k \) \((d_k \neq 0, \ 0 \leq k \leq n)\) has all its zeros in the annulus \( C = \{ z : r_1 \leq |z| \leq r_2 \} \), where

\[ r_1 = \min_{1 \leq k \leq n} \left\{ \frac{L_k}{L_{n+2} - 3} \left| \frac{d_0}{d_k} \right| \right\}^{\frac{1}{k}} \]  

(13)

and

\[ r_2 = \max_{1 \leq k \leq n} \left\{ \frac{L_{n+2} - 3}{L_k} \left| \frac{d_{n-k}}{d_n} \right| \right\}^{\frac{1}{k}}. \]  

(14)

**Corollary 1.2** Let \( C_k \) be the \( k \)th Catalan number defined by \( C_k = \frac{C(2k,k)}{k+1} \). Then a complex polynomial \( P(z) = \sum_{k=0}^n d_k z^k \) \((d_k \neq 0, \ 0 \leq k \leq n)\) has all its zeros in the annulus \( C = \{ z : r_1 \leq |z| \leq r_2 \} \), where

\[ r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C_{k-1}C_{n-k}}{C_n} \left| \frac{d_0}{d_k} \right| \right\}^{\frac{1}{k}} \]  

(15)

and

\[ r_2 = \max_{1 \leq k \leq n} \left\{ \frac{C_n}{C_{k-1}C_{n-k}} \left| \frac{d_{n-k}}{d_n} \right| \right\}^{\frac{1}{k}}. \]  

(16)
Theorem 1.1 implies that infinitely many annuli containing all the zeros of a complex polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) \((d_k \neq 0, \ 0 \leq k \leq n)\) can be obtained from infinitely many sets of positive numbers \( \{A_k\}_{k=1}^{n} \) such that \( \sum_{k=1}^{n} A_k = 1 \). Then it is natural to ask what set of positive numbers \( \{A_k\}_{k=1}^{n} \) with \( \sum_{k=1}^{n} A_k = 1 \) yields the best result. In this paper an answer to such a question is given. In other words we identify the smallest annulus \( A^* = \{r_1^* \leq |z| \leq r_2^*\} \) containing all the zeros of \( P(z) \) by computing the following two quantities:

\[
r_1^* = \max_{\sum_{k=1}^{n} A_k = 1 \ A_k > 0} \min_{1 \leq k \leq n} \left\{ A_k \left| \frac{d_0}{d_k} \right| \right\}^{\frac{1}{n}} \tag{17}
\]

and

\[
r_2^* = \min_{\sum_{k=1}^{n} A_k = 1 \ A_k > 0} \max_{1 \leq k \leq n} \left\{ \frac{1}{A_k} \left| \frac{d_{n-k}}{d_n} \right| \right\}^{\frac{1}{n}} \tag{18}
\]

It is noted that \( r_1^* \) and \( r_2^* \) can be computed independently each other.

## 2 Main results

**Theorem 2.1** Let \( r_1^* \) and \( r_2^* \) be as defined in (17) and (18). Then a complex polynomial \( P(z) = \sum_{k=0}^{n} d_k z^k \) \((d_k \neq 0, \ 0 \leq k \leq n)\) has all its zeros in the annulus \( A^* = \{z : r_1^* \leq |z| \leq r_2^*\} \), where

\[
r_1^* = \frac{|d_0| p}{|d_1|} \tag{19}
\]

and

\[
r_2^* = \frac{|d_{n-1}|}{|d_n| q}. \tag{20}
\]

Here \( p \) is the unique positive root of the equation

\[
\frac{|d_n||d_0|^{n-1}}{|d_1|^n} x^n + \frac{|d_{n-1}||d_0|^n-2}{|d_1|^{n-1}} x^{n-1} + \cdots + \frac{|d_2||d_0|}{|d_1|^2} x^2 + x - 1 = 0 \tag{21}
\]

and \( q \) is the unique positive root of the equation

\[
\frac{|d_0||d_n|^{n-1}}{|d_{n-1}|^n} x^n + \frac{|d_1||d_n|^{n-2}}{|d_{n-1}|^{n-1}} x^{n-1} + \cdots + \frac{|d_2||d_n|}{|d_{n-1}|^2} x^2 + x - 1 = 0. \tag{22}
\]

**Proof.** It is easily seen that each \( \left\{ A_k \left| \frac{d_0}{d_k} \right| \right\}^{\frac{1}{n}} \) in (17) is a concave function of \( A_k \). Then \( r_1^* \) is attained at the intersection of \( n \) concave functions, i.e.,

\[
A_1 \left| \frac{d_0}{d_1} \right| = \left\{ A_2 \left| \frac{d_0}{d_2} \right| \right\}^{\frac{1}{2}} = \cdots = \left\{ A_k \left| \frac{d_0}{d_k} \right| \right\}^{\frac{1}{k}} = \cdots = \left\{ A_n \left| \frac{d_0}{d_n} \right| \right\}^{\frac{1}{n}} \tag{23}
\]
from which we have
\[ A_k = A_1^k \frac{|d_k| |d_0|^{k-1}}{|d_1|^k}, \quad 2 \leq k \leq n. \] (24)

Substituting (24) into the equation \( \sum_{k=1}^{n} A_k = 1 \) yields
\[ \frac{|d_n||d_0|^{n-1}}{|d_1|^n} A_1^n + \frac{|d_{n-1}||d_0|^{n-2}}{|d_1|^{n-1}} A_1^{n-1} + \cdots + \frac{|d_2||d_0|}{|d_1|^2} A_1^2 + A_1 - 1 = 0. \] (25)

Let \( p \) denote the unique positive root of (25). Then \( r_1^* \) is given by (19).

Similarly each \( \left\{ \frac{1}{A_k} \frac{|d_n-1|}{|d_n|} \right\}^\frac{1}{k} \) in (18) is a convex function of \( A_k \). Then \( r_2^* \) is attained at the intersection of \( n \) convex functions, i.e.,
\[ \frac{1}{A_1} \frac{|d_{n-1}|}{|d_n|} = \left( \frac{1}{A_2} \frac{|d_{n-2}|}{|d_n|} \right)^\frac{1}{2} = \cdots = \left( \frac{1}{A_k} \frac{|d_{n-k}|}{|d_n|} \right)^\frac{1}{k} = \cdots = \left( \frac{1}{A_1} \frac{|d_0|}{|d_n|} \right)^\frac{1}{n}. \] (26)

from which we obtain
\[ A_k = A_1^k \frac{|d_{n-k}| |d_n|^{k-1}}{|d_{n-1}|^k}, \quad 2 \leq k \leq n. \] (27)

Substituting (27) into the equation \( \sum_{k=1}^{n} A_k = 1 \) yields
\[ \frac{|d_0||d_n|^{n-1}}{|d_{n-1}|^n} A_1^n + \frac{|d_1||d_n|^{n-2}}{|d_{n-1}|^{n-1}} A_1^{n-1} + \cdots + \frac{|d_{n-2}||d_n|}{|d_{n-1}|^2} A_1^2 + A_1 - 1 = 0. \] (28)

Let \( q \) denote the unique positive root of (28). Then \( r_2^* \) is given by (20).

### 3 Computations

**Example 3.1** [5] Consider a polynomial given by
\[ P(z) = z^3 + 0.1z^2 + 0.1z + 0.7. \]

The actual annulus containing all the zeros of \( P(z) \) is \( C_1 = \{ z : 0.8840 \leq |z| \leq 0.8899 \} \) with the area of 0.0328. According to [5], Corollary 1.1 gives the annulus \( C_2 = \{ z : 0.7047 \leq |z| \leq 1.1186 \} \) with the area of 2.3701. On the other hand, Theorem 2.1 yields the annulus \( C_3 = \{ z : 0.8197 \leq |z| \leq 0.9614 \} \) with the area of 0.7929. As is expected, Theorem 2.1 gives a much better result than that of Corollary 1.1 with 67% improvement in the area of annulus.

**Example 3.2** [5] Consider a polynomial given by
\[ P(z) = z^4 + 0.01z^3 + 0.1z^2 + 0.2z + 0.4. \]
The actual annulus for this example is given by $C_1 = \{z : 0.7190 \leq |z| \leq 0.8801\}$ with the area of 0.8093. As shown in [5], Corollary 1.2 gives the annulus $C_2 = \{z : 0.6147 \leq |z| \leq 1.1186\}$ with the area of 2.7427. The annulus obtained from Theorem 2.1 is $C_3 = \{z : 0.6805 \leq |z| \leq 0.9048\}$ with the area of 1.1172. Again Theorem 2.1 yields a much better result than that of Corollary 1.2 with 59% improvement in the area of annulus.

Example 3.2 [5] Consider a polynomial given by

$$P(z) = z^5 + 0.006z^4 + 0.01z^3 + 0.2z^2 + 0.3z + 1.$$ 

The actual annulus for $P(z)$ is $C_1 = \{z : 0.9526 \leq |z| \leq 1.0607\}$ with the area of 0.6873. According to [5], the annulus obtained from Corollary 1.2 is $C_2 = \{z : 0.7715 \leq |z| \leq 1.2805\}$ with the area of 3.2801. On the other hand, Theorem 2.1 yields the annulus $C_3 = \{z : 0.8914 \leq |z| \leq 1.0975\}$ with the area of 1.2876, which is an improvement of 61% over the area obtained from Corollary 1.2.

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References


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