Polynomial Representation and Degree Sequence of a Graph

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Abstract

If $G = (V(G), E(G))$ is a graph and $\Delta(G)$ is the maximum degree of $G$, the polynomial $f_G(x) = \sum_{i=1}^{n} a_i x^i$, where $a_i$ is the number of vertices of $G$ having degree $i$ for each $i = 1, 2, \cdots n = \Delta(G)$, is called the polynomial representation of $G$. If $\langle d_1, d_2, \cdots, d_p \rangle$ is the degree sequence of graph $G$, where $d_1 \leq d_2 \leq \cdots \leq d_p$ and $p$ is the order of $G$, then $f_G(x) = \sum_{i=1}^{p} x^{d_i}$. In this paper we give the polynomial representation and describe the degree sequence of the join, corona, lexicographic product, Cartesian product, and Tensor product of two graphs.

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1 Introduction

Let $G = (V(G), E(G))$ be a connected graph and $v \in V(G)$. The neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of $v \in V(G)$, denoted by $deg_G(v)$, is equal to the cardinality of $N_G(v)$. The
maximum degree of $G$, denoted by $\Delta(G)$, is equal to $\max \{deg_G(v) : v \in V(G) \}$. Suppose $\Delta(G) = n$. For each $i = 1, 2, \cdots, n$, let $a_i$ be the number of vertices of $G$ with degree $i \geq 0$. Then the polynomial $f_G(x) = \sum_{i=1}^{n} a_i x^i$ is called the polynomial representation of $G$. Equivalently, $f_G(x) = \sum_{v \in V(G)} x^{N_G(v)}$. While every graph $G$ has a polynomial representation, it is not true that a polynomial is always a polynomial representation of some simple graph (e.g. there is no simple graph $G$ with $f_G(x) = 2x^3 + 1$). A polynomial $f(x)$ is said to be graphic if there exists a simple graph $G$ such that $f_G(x) = f(x)$.

Now, if $p$ is the order of $G$ and $d_1, d_2, \cdots, d_p$ are degrees of the vertices $G$, where $d_1 \geq d_2 \geq \cdots \geq d_p$, then we refer to the sequence $\langle d_1, d_2, \cdots, d_p \rangle$ as the degree sequence of $G$. It is easy to see that $f_G(x) = \sum_{i=1}^{p} x^{d_i}$. As defined in [2], a sequence $\langle d_1, d_2, \cdots, d_p \rangle$ of nonnegative integers, where $d_1 \geq d_2 \geq \cdots \geq d_p$, is graphic if there exists a simple graph $G$ with degree sequence $\langle d_1, d_2, \cdots, d_p \rangle$. The degree sequence of a graph had been extensively studied and investigated by various authors (see [1], [3], [4], [5], and [6]). In particular, some characterizations had been obtained as to when a given non-increasing sequence of non-negative integers graphical.

This paper introduces for the first time the concept of polynomial representation of a graph. As shown in this paper, the polynomial representation of a graph makes it easier to describe the degree sequence of the join, corona, lexicographic product, Cartesian product, and Tensor product of two graphs. We also state, as a direct consequence of the characterization obtained by Erdős and Gallai in [3], necessary and sufficient condition for a given polynomial to be graphic.

### 2 Graphical Polynomials

We first state the following known result on the degree sequence of a graph.

**Theorem 2.1 ([3])** A sequence $\langle d_1, d_2, \cdots, d_p \rangle$ of nonnegative integers, where $d_1 \geq d_2 \geq \cdots \geq d_p$, is graphic (or graphic) if and only if its sum is even and for all $k = 1, 2, \cdots, p$, we have

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} \min \{k, d_i\}.$$ 

Consider a polynomial $f(x) = \sum_{i=1}^{q} a_i x^{k_i}$, where $a_i$ and $k_i$ are positive integers for all $i = 1, 2, \cdots, q$ and $k_1 > k_2 > \cdots > k_q$. For each $j$, with
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$1 \leq j \leq \sum_{i=1}^{q} a_i = p$, let $d_j = k_i$ whenever $\sum_{r=0}^{i-1} a_r < j \leq \sum_{r=0}^{i} a_r$ for $i = 1, 2, \ldots, q$, where $a_0 = 0$. Clearly, $\langle d_1, d_2, \ldots, d_p \rangle$ is a sequence of positive integers and $f(x) = \sum_{i=1}^{p} x^{d_i}$. Henceforth, the sequence $\langle d_1, d_2, \ldots, d_p \rangle$ will be referred to as the sequence induced by $f$. The first result characterizes all graphical polynomials. The proof follows from the definition of graphical polynomial and Theorem 2.1 by Erdős and Gallai in [3].

**Theorem 2.2** Let $f(x) = \sum_{i=1}^{q} a_i x^{k_i}$ be a polynomial, where $a_i$ and $k_i$ are positive integers for all $i = 1, 2, \ldots, q$ and $k_1 > k_2 > \cdots > k_q$, and let $\langle d_1, d_2, \ldots, d_p \rangle$ be the sequence induced by $f$. Then the following statements are equivalent:

(a) The polynomial $f(x)$ is graphic.

(b) $\sum_{i=1}^{k} d_i$ is even and for all $k = 1, 2, \ldots, p$, we have

$$
\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} \min\{k, d_i\}.
$$

**3 Join of Graphs**

The join $G + H$ of two graphs $G$ and $H$ is the graph with vertex set

$$
V(G + H) = V(G) \cup V(H)
$$

and edge set

$$
E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.
$$

**Theorem 3.1** Let $G$ and $H$ be graphs with polynomial representations $f_G(x)$ and $f_H(x)$ and orders $p$ and $q$, respectively. Then

$$
f_{G+H}(x) = x^p f_G(x) + x^q f_H(x).
$$

**Proof:** Let $v \in V(G + H)$. If $v \in V(G)$, then $N_{G+H}(v) = N_G(v) \cup V(H)$. If $v \in V(H)$, then $N_{G+H}(v) = N_H(v) \cup V(G)$. It follows that
\[
\begin{align*}
f_{G+H}(x) &= \sum_{v \in V(G+H)} x^{\left|N_{G+H}(v)\right|} \\
&= \sum_{v \in V(G)} x^{\left|N_{G+H}(v)\right|} + \sum_{v \in V(H)} x^{\left|N_{G+H}(v)\right|} \\
&= \sum_{v \in V(G)} x^{\left|N_G(v)\right|+\left|V(H)\right|} + \sum_{v \in V(H)} x^{\left|N_H(v)\right|+\left|V(G)\right|} \\
&= x^q \sum_{v \in V(G)} x^{\left|N_G(v)\right|} + x^p \sum_{v \in V(H)} x^{\left|N_H(v)\right|} \\
&= x^q f_G(x) + x^p f_H(x).
\end{align*}
\]

The following results are immediate from the above result:

**Corollary 3.2** Let \( G \) and \( H \) be graphs of orders \( p \) and \( q \), respectively. If \( H \) is \( r \)-regular, then
\[
f_{G+H}(x) = x^q f_G(x) + qx^{p+r}.
\]
In particular, \( f_{G+K_q}(x) = x^q f_G(x) + qx^{p+q-1} \).

**Corollary 3.3** Let \( m \) and \( n \) be positive integers. Then
\[
f_{K_{m,n}}(x) = mx^n + nx^m.
\]

\(|V(H)| = n \) and \( f_G(x) = m \), and \( f_H(x) = n \). The desired result now follows from Theorem 3.1.

**Theorem 3.4** Let \( G \) and \( H \) be connected graphs with degree sequences \( \langle d_1, d_2, \cdots, d_p \rangle \) and \( \langle r_1, r_2, \cdots, r_q \rangle \), respectively. Then the terms of the degree sequence of \( G + H \) are the elements of the set \( \{q + d_i : 1 \leq i \leq p\} \cup \{p + r_i : 1 \leq i \leq q\} \).

**Proof:** The polynomial representations of \( G \) and \( H \) are, respectively, \( f_G(x) = \sum_{i=1}^{p} x^{d_i} \) and \( f_H(x) = \sum_{i=1}^{q} x^{r_i} \). By Theorem 3.1,
\[
f_{G+H}(x) = x^q \sum_{i=1}^{p} x^{d_i} + x^p \sum_{i=1}^{q} x^{r_i} \\
= \sum_{i=1}^{p} x^{q+d_i} + \sum_{i=1}^{q} x^{p+r_i}.
\]
It follows that the terms of the degree sequence of \( G + H \) are the elements of the set \( \{q + d_i : 1 \leq i \leq p\} \cup \{p + r_i : 1 \leq i \leq q\} \).
4 Corona of Graphs

The corona $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex in the $i$th copy of $H$. For every $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$, $v \in V(G)$.

**Theorem 4.1** Let $G$ be a connected graph of order $p$ and $H$ any graph of order $q$. Then

$$f_{G \circ H}(x) = x^q f_G(x) + px f_H(x).$$

**Proof:** Let $v \in V(G \circ H)$. If $v \in V(G)$, then $N_{G \circ H}(v) = N_G(v) \cup V(H^v)$. If $v \in V(H^u)$ for some $u \in V(G)$, then $N_{G \circ H}(v) = N_{H^u}(v) \cup \{u\}$. It follows that

$$f_{G \circ H}(x) = \sum_{v \in V(G \circ H)} x^{|N_{G \circ H}(v)|}$$

$$= \sum_{v \in V(G)} x^{|N_{G \circ H}(v)|} + \sum_{v \in V(G \circ H) \setminus V(G)} x^{|N_{G \circ H}(v)|}$$

$$= \sum_{v \in V(G)} x^{|N_G(v)|+q} + \sum_{v \in V(G \circ H) \setminus V(G)} x^{|N_{H^v}(v)|+1}$$

$$= x^q \sum_{v \in V(G)} x^{|N_G(v)|} + px \sum_{v \in V(H)} x^{|N_{H^v}(v)|}$$

$$= x^q f_G(x) + px f_H(x).$$

The following result is immediate from the above result:

**Corollary 4.2** Let $G$ be a connected graph of order $p$ and $H$ an $r$-regular graph of order $q$. Then

$$f_{G \circ H}(x) = x^q f_G(x) + pq x^{r+1}.$$  

In particular, $f_{G \circ K_q}(x) = x^q (f_G(x) + pq)$.

**Theorem 4.3** Let $G$ be a connected graph and $H$ a graph with degree sequences $\langle d_1, d_2, \ldots, d_p \rangle$ and $\langle r_1, r_2, \ldots, r_q \rangle$, respectively. Then the terms of the degree sequence of $G \circ H$ are the elements of the set $\{q + d_i : 1 \leq i \leq p\} \cup \{1 + r_i : 1 \leq i \leq q\}$, where $p$ consecutive terms of the degree sequence are $1 + r_i$ for each $i$ with $1 \leq i \leq q$. 
Proof: The polynomial representations of \( G \) and \( H \) are, respectively, 
\[
    f_G(x) = \sum_{i=1}^{p} x^{d_i} \quad \text{and} \quad f_H(x) = \sum_{i=1}^{q} x^{r_i}.
\]
By Theorem 3.1, 
\[
    f_{G \circ H}(x) = x^q \sum_{i=1}^{p} x^{d_i} + px \sum_{i=1}^{q} x^{r_i} = \sum_{i=1}^{p} x^{q+d_i} + p \sum_{i=1}^{q} x^{1+r_i}.
\]
It follows that the terms of the degree sequence of \( G \circ H \) are the elements of the set \( \{q + d_i : 1 \leq i \leq p\} \cup \{1 + r_i : 1 \leq i \leq q\} \). Moreover, \( p \) consecutive terms of the degree sequence are \( 1 + r_i \) for each \( i \) with \( 1 \leq i \leq q \).

5 Lexicographic Product of Graphs

The lexicographic \( G[H] \) of two graphs \( G \) and \( H \) is the graph with \( V(G[H]) = V(G) \times V(H) \) and \((u, u')(v, v') \in E(G[H])\) if and only if either \( uv \in E(G) \) or \( u = v \) and \( u'v' \in E(H) \).

Theorem 5.1 Let \( G \) and \( H \) be connected graphs of orders \( m \) and \( n \), respectively. Then

\[
    f_{G[H]}(x) = f_G(x^n)f_H(x).
\]

Proof: Let \((a, b) \in V(G[H])\). Then \( N_{G[H]}((a, b)) = D \cup E \), where \( D = \{(u, v) \in V(G[H]) : au \in E(G)\} \) and \( E = \{(u, v) \in V(G[H]) : a = u \ \text{and} \ bv \in E(H)\} \). Thus

\[
    |N_{G[H]}((a, b))| = |D| + |E| = |V(H)||N_G(a)| + |N_H(b)|.
\]
Therefore
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\[ f_{G[H]}(x) = \sum_{(a,b) \in V(G[H])} x^{\left| N_{G[H]}((a,b)) \right|} \]

\[ = \sum_{(a,b) \in V(G[H])} x^{\left| V(H) \right| N_{G}(a) + \left| N_{H}(b) \right|} \]

\[ = \sum_{a \in V(G)} \sum_{b \in V(H)} x^{\left| V(H) \right| N_{G}(a)} \sum_{b \in V(H)} x^{\left| N_{H}(b) \right|} \]

\[ = f_{G}(x^{\left| V(H) \right|}) f_{H}(x) \]

\[ = f_{G}(x^{n}) f_{H}(x). \]

The following result is a direct consequence of Theorem 5.1:

**Corollary 5.2** Let \( G \) be a connected graph of order \( m \) and \( K_n \) the complete graph of order \( n \). Then

\[ f_{G[K_n]}(x) = n x^{n-1} f_{G}(x^{n}). \]

**Theorem 5.3** Let \( G \) and \( H \) be connected graphs with degree sequences \( \langle d_1, d_2, \cdots, d_p \rangle \) and \( \langle r_1, r_2, \cdots, r_q \rangle \), respectively. Then the terms of the degree sequence of \( G[H] \) are the elements of the set \( \{ qd_i + r_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q \} \).

**Proof:** The polynomial representations of \( G \) and \( H \) are, respectively, \( f_{G}(x) = \sum_{i=1}^{p} x^{d_i} \) and \( f_{H}(x) = \sum_{i=1}^{q} x^{r_i} \). By Theorem 4.1,

\[ f_{G[H]}(x) = \sum_{i=1}^{p} (x^{q})^{d_i} \sum_{i=1}^{q} x^{r_i} \]

\[ = \sum_{i=1}^{p} \sum_{j=1}^{q} x^{qd_i + r_j}. \]

It follows that the terms of the degree sequence of \( G[H] \) are the elements of the set \( \{ qd_i + r_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q \} \).

**Corollary 5.4** Let \( G \) be connected graph with degree sequence \( \langle d_1, d_2, \cdots, d_p \rangle \) and \( K_q \) the complete graph of order \( q \). Then the degree sequence of \( G[K_q] \) is \( \langle qd_1 + q - 1, qd_1 + q - 1, \cdots, qd_1 + q - 1, \]

\[ qd_2 + q - 1, \cdots, qd_2 + q - 1, \cdots, qd_p + q - 1, \cdots, qd_p + q - 1 \rangle. \]
6 Cartesian Product of Graphs

The Cartesian product \( G \times H \) of two graphs \( G \) and \( H \) is the graph with \( V(G \times H) = V(G) \times V(H) \) and \((u, u')(v, v') \in E(G \times H) \) if and only if either \( uv \in E(G) \) and \( u' = v' \) or \( u = v \) and \( u'v' \in E(H) \).

**Theorem 6.1** Let \( G \) and \( H \) be connected graphs. Then

\[
f_{G \times H}(x) = f_G(x)f_H(x).
\]

**Proof:** Let \((a, b) \in V(G \times H)\). Then \( N_{G \times H}((a, b)) = D \cup E \), where \( D = \{(u, v) \in V(G \times H) : b = v \text{ and } au \in E(G) \} \) and \( E = \{(u, v) \in V(G \times H) : a = u \text{ and } bv \in E(H) \} \). Thus

\[
|N_{G \times H}((a, b))| = |D| + |E| = |N_G(a)| + |N_H(b)|.
\]

Therefore

\[
f_{G \times H}(x) = \sum_{(a, b) \in V(G \times H)} x^{|N_{G \times H}((a, b))|} = \sum_{(a, b) \in V(G \times H)} x^{|N_G(a)| + |N_H(b)|} = \sum_{a \in V(G)} \sum_{b \in V(H)} x^{|N_G(a)|} x^{|N_H(b)|} = f_G(x)f_H(x).
\]

**Theorem 6.2** Let \( G \) and \( H \) be connected graphs with degree sequences \( \langle d_1, d_2, \cdots, d_p \rangle \) and \( \langle r_1, r_2, \cdots, r_q \rangle \), respectively. Then the terms of the degree sequence of \( G \times H \) are the elements of the set \( \{d_i + r_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\} \).

**Proof:** The polynomial representations of \( G \) and \( H \) are, respectively, \( f_G(x) = \sum_{i=1}^{p} x^{d_i} \) and \( f_H(x) = \sum_{i=1}^{q} x^{r_i} \). By Theorem 5.1, \( f_{G \times H}(x) = \sum_{i=1}^{p} x^{d_i} \sum_{i=1}^{q} x^{r_i} = \sum_{i=1}^{p} \sum_{j=1}^{q} x^{d_i + r_j} \). It follows that the terms of the degree sequence of \( G \times H \) are the elements of the set \( \{d_i + r_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\} \).
Corollary 6.3 Let $G$ be connected graph with degree sequence $(d_1, d_2, \ldots, d_p)$ and $K_q$ the complete graph of order $q$. Then the degree sequence of $G \times K_q$ is $\langle d_1 + q - 1, d_1 + q - 1, \ldots, d_1 + q - 1, d_2 + q - 1, \ldots, d_p + q - 1, d_p + q - 1 \rangle$.

7 Tensor Product of Graphs

The Tensor product of graphs $G$ and $H$ is the graph $G \otimes H$ with $V(G \otimes H) = V(G) \times V(H)$ and $(a, b)(u, v) \in E(G \otimes H)$ if and only if $au \in E(G)$ and $bv \in E(H)$.

Theorem 7.1 Let $G$ and $H$ be connected graphs with polynomial representations $f_G(x)$ and $f_H(x)$ and orders $p$ and $q$, respectively. Then

$$f_{G \otimes H}(x) = \sum_{a \in V(G)} f_H(x^{\left|N_G(a)\right|}) = \sum_{b \in V(H)} f_G(x^{\left|N_H(b)\right|}).$$

Proof: Let $(a, b) \in V(G \otimes H)$. By definition,

$$N_{G \otimes H}((a, b)) = \{(u, v) : au \in E(G) \text{ and } bv \in E(H)\}.$$

In other words,

$$N_{G \otimes H}((a, b)) = V(G \otimes H) \setminus (D \cup E),$$

where $D = \{(u, v) : au \notin E(G)\}$ and $E = \{(u, v) : bv \notin E(H)\}$. Note that $|D| = q(p - |N_G(a)|)$ and $|E| = p(q - |N_H(b)|)$. Now, since $((V(G) \setminus N_G(a)) \times (V(H) \setminus N_H(b))) = D \cap E$, it follows that

$$|N_{G \otimes H}((a, b))| = pq - [q(p - |N_G(a)|) + p(q - |N_H(b)|) - (p - |N_G(a)|)(q - |N_H(b)|)],$$

i.e., $|N_{G \otimes H}((a, b))| = |N_G(a)| \cdot |N_H(b)|$. By definition,

$$f_{G \otimes H}(x) = \sum_{(a, b) \in V(G \otimes H)} x^{\left|N_G(a)\right| \cdot \left|N_H(b)\right|} = \sum_{a \in V(G)} \sum_{b \in V(H)} (x^{\left|N_G(a)\right| \cdot \left|N_H(b)\right|}).$$

Therefore

$$f_{G \otimes H}(x) = \sum_{a \in V(G)} f_H(x^{\left|N_G(a)\right|}) = \sum_{b \in V(H)} f_G(x^{\left|N_H(b)\right|}).$$
Corollary 7.2 Let $G$ be a connected graph with polynomial representation $f_G(x)$. If $H$ is an $r$-regular connected graph ($r \geq 1$) of order $q$, then

$$f_{G \otimes H}(x) = qf_G(x^r).$$

In particular, if $H = K_q$, then $f_{G \otimes H}(x) = qf_G(x^{q-1})$.

Theorem 7.3 Let $G$ and $H$ be connected graphs with degree sequences $\langle d_1, d_2, \cdots, d_p \rangle$ and $\langle r_1, r_2, \cdots, r_q \rangle$, respectively. Then the terms of the degree sequence of $G \otimes H$ are the elements of the set $\{d_ir_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$.

Proof: The polynomial representations of $G$ and $H$ are, respectively, $f_G(x) = \sum_{i=1}^{p} x^{d_i}$ and $f_H(x) = \sum_{i=1}^{q} x^{r_i}$. By Theorem 7.1,

$$f_{G \otimes H}(x) = \sum_{i=1}^{p} \sum_{j=1}^{q} (x^{d_i})^{r_j} = \sum_{i=1}^{p} \sum_{j=1}^{q} x^{d_ir_j}.$$

It follows that the terms of the degree sequence of $G \otimes H$ are the elements of the set $\{d_ir_j : 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}$.

The following result follows from Theorem 7.3 (also from Corollary 7.2).

Corollary 7.4 Let $G$ be a connected graph with degree sequence $\langle d_1, d_2, \cdots, d_p \rangle$. If $H$ is an $r$-regular connected graph ($r \geq 1$) of order $q$, then the degree sequence of $G \otimes H$ is $\langle rd_1, rd_1, \cdots, rd_1, \underbrace{rd_2, rd_2, \cdots, rd_2}_{\text{q terms}}, \underbrace{rd_p, rd_p, \cdots, rd_p}_{\text{q terms}} \rangle$.

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