Differential of a Graph and Its Relation to the Concept of Domination

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Abstract

Let $G = (V(G), E(G))$ be an arbitrary graph. For $X \subseteq V(G)$, the boundary $B(X)$ of $X$ is the set of vertices in $V(G) \setminus X$ that have a neighbor in $X$ and the differential of $X$ is $\partial(X) = |B(X)| - |X|$. The differential $\partial(G)$ of graph $G$ is $\partial(G) = \max\{\partial(X) : X$ is a subset of $V(G)\}$. It is shown in this paper that for any graph $G$ of order $n \geq 2$, $\partial(G)$ is between 0 and $(n - 2)$. Graphs whose differential is $|V(G)| - 2$ are characterized. Differentials of the join, corona, and lexicographic product of two graphs are also determined. For the lexicographic product $G[H]$ of two graphs, it is shown that if the order of $H$ is $|V(H)| - 2$, then one half of the difference of the order of $G[H]$ and its differential is equal to its domination number. Otherwise, this difference is the total domination of $G[H]$.

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1 Introduction

Let $G = (V(G), E(G))$ be an arbitrary graph of order $n$. In [1], the following game was introduced: Suppose you are allowed to buy as many tokens as you
like, say \( k \leq n \) tokens, at a cost of one dollar each. You then place the tokens on some subset of \( V(G) \) consisting of \( k \) vertices (or nodes). For each vertex of \( G \) which has no token on it, but is adjacent to a vertex with a token on it, you receive one dollar. The main goal of this game is to maximize your profit, that is, the total value received minus the total cost of the tokens bought. Observed that in this game one does not get any credit for the vertices on which he or she places a token. This game and its underlying theory was introduced by Slater et al. in [1].

The **neighborhood** of a vertex \( v \) of \( G \) is the set \( N_G(v) = \{u \in V(G) : uv \in E(G)\} \). For a set \( X \subseteq V(G) \), the **neighborhood** of \( X \) is the set \( N(X) = \bigcup_{v \in X} N(v) \). The **closed neighborhood** of \( X \) is the set \( N[X] = N(X) \cup X \). The **boundary** of \( X \), denoted by \( B(X) \), is the set \( (V(G) \setminus X) \cap N(X) \). The **\( B \)-differential** of \( X \) is \( |B(X)| \). The \( B \)-differential of \( G \) is \( \Psi(G) = \max\{|B(X)| : X \subseteq V(G)\} \). The **differential** \( \partial(X) \) of \( X \) is given by \( \partial(X) = |B(X)| - |X| \). The **differential of a graph** \( G \) is given by \( \partial(G) = \max\{\partial(X) : X \subseteq V(G)\} \).

A set \( X \subseteq V(G) \) is a **dominating set** of \( G \) if \( N[X] = V(G) \). The **domination number** \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set. If \( X \) is a dominating set with \( |X| = \gamma(G) \), then we call \( X \) a **minimum dominating set of** \( G \). If \( N(X) = V(G) \), then we say that \( X \) is a **total dominating set** of \( G \). The **total domination number** \( \gamma_t(G) \) of \( G \) is the minimum cardinality of a total dominating set. If \( X \) is a total dominating set with \( |X| = \gamma_t(G) \), then we call \( X \) a **minimum total dominating set of** \( G \).

The sets \( B(X) \) were considered by Slater in [3]. The parameter \( \partial(G) \) was also considered by Goddard and Henning in [2].

It should be noted that the parameter \( \partial(G) \) corresponds to the maximum profit a player obtains in playing the aforementioned game in graph \( G \). It is shown that \( 0 \leq \partial(G) \leq n - 2 \) for any graph \( G \) of order \( n \geq 2 \). Graphs which yield a maximum profit of \( n - 2 \) are also characterized.

## 2 Results

**Lemma 2.1** Let \( G \) be any graph of order \( n \geq 2 \). Then \( 0 \leq \partial(G) \leq n - 2 \).

**Proof:** Let \( X = \emptyset \). Then \( B(X) = \emptyset \); hence \( \partial(X) = 0 \leq \partial(G) \). Let \( Y \) be a subset of \( V(G) \) with \( \partial(G) = \partial(Y) \). If \( Y = \emptyset \), then \( \partial(Y) = 0 \leq n - 2 \). If \( Y \neq \emptyset \), then \( |Y| \geq 1 \) and \( |V(G) \setminus Y| \leq n - 1 \). Therefore \( \partial(G) = \partial(Y) = |B(Y)| - |Y| \leq (n - 1) - 1 = n - 2 \). Accordingly, \( 0 \leq \partial(G) \leq n - 2 \). \( \Box \)

The **degree** of vertex \( v \) of graph \( G \) is given by \( \deg_G(v) = |N(v)| \). The **maximum degree** of \( G \), denoted by \( \Delta(G) \), is \( \max\{\deg_G(v) : v \in V(G)\} \). If \( v \) is a vertex of \( G \) with \( \deg_G(v) = \Delta(G) \), then we call \( v \) a **vertex of maximum degree** in \( G \).

The following result is due to Lewis et al. [L].
Theorem 2.2  For any graph \( G \), \( \Delta(G) - 1 \leq \partial(G) \).

The next result says that if one is playing on a connected graph, only at most a break even game can be achieved in the trivial and the complete graph of order two.

Theorem 2.3  Let \( G \) be a graph of order \( n \). Then \( \partial(G) = 0 \) if and only if the components of \( G \) are \( K_1 \) or \( K_2 \) (or both).

Proof: Suppose \( \partial(G) = 0 \). Then \( \Delta(G) = 0 \) or \( \Delta(G) = 1 \) by Theorem 2.2. This implies that every component of \( G \) is \( K_1 \) or \( K_2 \).

The converse is clear. \( \square \)

We now characterize those graphs \( G \) for which \( \partial(G) = n - 2 \).

Theorem 2.4  Let \( G \) be any graph of order \( n \geq 2 \). Then \( \partial(G) = n - 2 \) if and only if there exists a vertex \( v \) of \( G \) such that \( v \in N(x) \) for all \( x \in V(G) \setminus \{v\} \).

Proof: Suppose \( \partial(G) = n - 2 \) and let \( X \subseteq V(G) \) such that \( \partial(G) = \partial(X) \). If \( |X| \geq 2 \), then \( |V(G) \setminus X| \leq n - 2 \). Hence \( \partial(G) = \partial(X) = |B(X)| - |X| \leq (n - 2) - 2 = n - 4 \), contrary to our assumption that \( \partial(G) = n - 2 \). This implies that \( |X| = 1 \), say \( X = \{v\} \). If \( B(X) \neq V(G) \setminus \{v\} \), then \( \partial(G) < |V(G) \setminus \{v\}| = n - 2 \). This, again, gives us a contradiction. Thus \( B(X) = V(G) \setminus \{v\} \). Therefore \( v \in N(x) \) for all \( x \in V(G) \setminus \{v\} \).

For the converse, assume that there exists a vertex \( v \) of \( G \) such that \( v \in N(x) \) for all \( x \in V(G) \setminus \{v\} \). Set \( X = \{v\} \). Then \( B(X) = V(G) \setminus \{v\} \) and \( \partial(G) \geq n - 2 \). By Lemma 2.1, \( \partial(G) = n - 2 \). \( \square \)

The following result is a direct consequence of Theorem 2.4.

Corollary 2.5  \( \partial(K_n) = n - 2 \) for any integer \( n \geq 2 \).

The join \( G + H \) of two graphs \( G \) and \( H \) is the graph with vertex set \( V(G + H) = V(G) \cup V(H) \) and edge set \( E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\} \).

We now give the differential of the join two graphs.

Theorem 2.6  Let \( G \) and \( H \) be graphs of orders \( n \) and \( m \), respectively.

(i) If either \( G \) or \( H \) is complete, then \( \partial(G + H) = n + m - 2 \).

(ii) If \( G \) and \( H \) are non-complete, then

\[
\partial(G + H) = \max\{n + m - 4, \Delta(G) + m - 1, \Delta(H) + n - 1\}.
\]
Proof: (i) Assume that \( H = K_m \) and pick \( a \in V(H) \). By the adjacency in \( G + H \), it follows that \( v \in N_{G+H}(a) \) for all \( v \in V(G + H) \setminus \{a\} \). Hence by Theorem 2.4, \( \partial(G + H) = n + m - 2 \).

(ii) Let \( k = \max\{n + m - 4, \Delta(G) + m - 1, \Delta(H) + n - 1\} \). Consider the following cases:

Case 1. Assume that \( k = n + m - 4 \). Pick \( a \in V(G) \) and \( b \in V(H) \) and set \( X = \{a, b\} \). Then \( B(X) = V(G + H) \setminus X \) and \( \partial(X) = n + m - 4 \leq \partial(G + H) \).

Next, let \( Y \subseteq V(G + H) \) such that \( \partial(G + H) = \partial(Y) \). If \( Y \cap V(G) \neq \emptyset \) and \( Y \cap V(H) \neq \emptyset \), then \( B(Y) = V(G + H) \setminus Y \). Hence

\[
\partial(G + H) = \partial(Y) = n + m - 2|Y| \leq n + m - 4.
\]

Suppose now that either \( Y \subseteq V(G) \) or \( Y \subseteq V(H) \), say \( Y \subseteq V(G) \). Then \( B(Y) = V(H) \cup (N_G(Y) \setminus Y) \subseteq V(H) \cup (V(G) \setminus Y) \). If \( |Y| \geq 2 \), then,

\[
\partial(G + H) = \partial(Y) \leq n + m - 2|Y| \leq n + m - 4.
\]

If \( |Y| = 1 \), say \( Y = \{v\} \), then \( B(Y) = V(H) \cup N_G(v) \). Thus,

\[
\partial(G + H) = \partial(Y) \leq \Delta(G) + m - 1 \leq n + m - 4.
\]

Therefore,

\[
\partial(G + H) = n + m - 4.
\]

Case 2. Assume that \( k = \Delta(G) + m - 1 \). Let \( v \in V(G) \) with \( \deg_G(v) = \Delta(G) \). Set \( X = \{v\} \). Then \( B(X) = N_G(v) \cup V(H) \) and \( \partial(X) = \Delta(G) + m - 1 \leq \partial(G + H) \).

Next, let \( Y \subseteq V(G + H) \) such that \( \partial(G + H) = \partial_{G+H}(Y) \). If \( Y \cap V(G) \neq \emptyset \) and \( Y \cap V(H) \neq \emptyset \), then \( B(Y) = V(G + H) \setminus Y \). Hence, \( \partial(G + H) = \partial_{G+H}(Y) \leq n + m - 4 \leq \Delta(G) + m - 1 \). Assume, without loss of generality, that \( Y \subseteq V(G) \).

Then \( B_{G+H}(Y) = V(H) \cup (N_G(Y) \setminus Y) \subseteq V(H) \cup (V(G) \setminus Y) \). If \( |Y| \geq 2 \), then \( \partial(G + H) = \partial(Y) \leq n + m - 2|Y| \leq n + m - 4 \leq \Delta(G) + m - 1 \). If \( |Y| = 1 \), say \( Y = \{w\} \), then \( B(Y) = V(H) \cup N_G(w) \). Thus, \( \partial(G + H) = \partial(Y) = |N_G(w)| + m - 1 \leq \Delta(G) + m - 1 \). Therefore, \( \partial(G + H) = \Delta(G) + m - 1 \).

Case 3. Assume that \( k = \Delta(H) + n - 1 \). Following the arguments of the proof of Case 2, we have \( \partial(G + H) = \Delta(H) + n - 1 \). \( \square \)

From Theorem 2.6(i), it follows that \( \partial(W_n) = \partial(S_n) = \partial(F_n) = n - 1 \), where \( W_n \), \( S_n \), and \( F_n \) are the wheel, star, and fan of order \( n + 1 \).

The next results, which are direct consequences of Theorem 2.6(ii), give the differentials of the complete bipartite, the generalized fan, and the generalized wheel.

**Corollary 2.7** Let \( n \geq 2 \) and \( m \geq 2 \) be integers. Then

\[
\partial(K_{m,n}) = \max\{m + n - 4, n - 1, m - 1\}.
\]
Corollary 2.8 Let \( n \geq 2 \) and \( m \geq 3 \) be integers. Then
\[
\partial(K_n + P_m) = \max\{n + m - 4, m - 1, n + 1\}.
\]

Corollary 2.9 Let \( n \geq 2 \) and \( m \geq 3 \) be integers. Then
\[
\partial(K_n + C_m) = \max\{n + m - 4, m - 1, n + 1\}.
\]

The corona \( G \circ H \) of two graphs \( G \) and \( H \) is the graph obtained by taking one copy of \( G \) of order \( n \) and \( n \) copies of \( H \), and then joining the \( i \)th vertex of \( G \) to every vertex in the \( i \)th copy of \( H \). We denote by \( v + H^v \) the subgraph of \( G \circ H \) obtained by joining the vertex \( v \in V(G) \) to every vertex of the copy \( H^v \) of \( H \).

Lemma 2.10. Let \( G \) and \( H \) be connected graphs. If \( X \subseteq V(G) \), then
\[
B_{G \circ H}(X) = B_G(X) \cup S \quad \text{and} \quad \partial_{G \circ H}(X) = |B_G(X)| + |X||V(H)| - 1,
\]
where \( S = \cup_{v \in X} V(H^v) \).

Proof: Let \( x \in B_{G \circ H}(X) \). Then \( x \in N_{G \circ H}(X) \) and \( x \notin X \). This implies that there exists \( y \in X \) such that \( x \in N_{G \circ H}(y) \setminus X \). Hence, \( x \in N_G(y) \setminus X \) or \( x \in V(H^y) \). Consequently, \( x \in B_G(X) \cup S \), showing that
\[
B_{G \circ H}(X) \subseteq B_G(X) \cup S.
\]
The reverse inclusion is clear. Therefore, \( B_{G \circ H}(X) = B_G(X) \cup S \). Moreover, by definition, we have
\[
\partial_{G \circ H}(X) = |B_G(X)| + |S| - |X|
= |B_G(X)| + |X||V(H)| - |X|
= |B_G(X)| + |X||V(H)| - 1.
\]

Lemma 2.11. Let \( G \) and \( H \) be connected graphs and \( Y \subseteq V(G \circ H) \) such that \( \partial_{G \circ H}(Y) = \partial(G \circ H) \). If \( v \in Y \cap V(G) \), then \( u \notin Y \) for all \( u \in V(H^v) \).

Proof: Suppose \( v \in Y \cap V(G) \) and suppose further that \( u \in Y \) for some \( u \in V(H^v) \). Let \( X = Y \setminus \{u\} \) and set \( A = \cup_{x \in X} (N_{G \circ H}(x) \setminus Y) \). Then
\[
B_{G \circ H}(Y) = A \cup (N_{G \circ H}(u) \setminus Y).
\]
Since \( (N_{G \circ H}(u) \setminus Y) \subseteq (N_{G \circ H}(v) \setminus Y) \), it follows that
\[
B_{G \circ H}(Y) = A \cup (N_{G \circ H}(v) \setminus Y)
= \cup_{x \in X} (N_{G \circ H}(x) \setminus Y).
\]
Thus,
\[ \partial_{G \circ H}(Y) = |\bigcup_{x \in X} N_{G \circ H}(x) \setminus Y| - |Y|. \]
Now, since \( N_{G \circ H}(x) \setminus Y \subseteq N_{G \circ H}(x) \setminus X \) for all \( x \in X \) and \( |X| < |Y| \), we have
\[ \partial_{G \circ H}(Y) < |\bigcup_{x \in X} N_{G \circ H}(x) \setminus X| - |X| = \partial_{G \circ H}(X). \]
This is a contradiction to the fact that \( \partial_{G \circ H}(Y) = \partial(G \circ H) \). Therefore, \( u \notin Y \) for all \( u \in V(H^v) \) whenever \( v \in Y \cap V(G) \). \( \square \)

**Lemma 2.12.** Let \( G \) be a connected graph. If \( Y \subseteq V(G \circ K_1) \) such that \( \partial_{G \circ K_1}(Y) = \partial(G \circ K_1) \), then \( \partial(G \circ K_1) = \partial_{G \circ K_1}(Y \cap V(G)) \).

**Proof:** Let \( A = Y \cap V(G) \) and \( E = Y \cap (V(G))^c \). By definition,
\[ B_{G \circ K_1}(A) = \bigcup_{v \in A} (N_{G \circ K_1}(v) \setminus A) \]
and
\[ B_{G \circ K_1}(Y) = [\bigcup_{v \in A} (N_{G \circ K_1}(v) \setminus Y)] \cup [\bigcup_{v \in E} (N_{G \circ K_1}(v) \setminus Y)]. \]
Let \( v \in E \). Then, there exists \( z \in V(G) \) such that \( v \in K_1^z \). By Lemma 2.11, \( z \notin Y \). Thus, \( z \in N_{G \circ K_1}(v) \setminus Y \). It follows that \( |N_{G \circ K_1}(v) \setminus Y| = 1 \) for every \( v \in E \) and so, \( |\bigcup_{v \in E} (N_{G \circ K_1}(v) \setminus Y)| = |E| \). Therefore,
\[ \partial_{G \circ K_1}(Y) = \partial(G \circ K_1) \]
\[ \leq |\bigcup_{v \in A} N_{G \circ K_1}(v) \setminus Y| + |E| - |Y| \]
\[ \leq |\bigcup_{v \in A} N_{G \circ K_1}(v) \setminus A| - |A| - \partial_{G \circ K_1}(A). \]
This shows that \( \partial_{G \circ K_1}(Y \cap V(G)) = \partial(G \circ K_1) \). \( \square \)

**Theorem 2.13.** Let \( G \) be a connected graph. Then \( \partial(G \circ K_1) = \Psi(G) \).

**Proof:** Let \( X \subseteq V(G) \) with \( |B_G(X)| = \Psi(G) \). From Lemma 2.10, we have \( \Psi(G) \leq \partial(G \circ K_1) \).
Next, let \( Y \subseteq V(G \circ K_1) \) with \( \partial(G \circ K_1) = \partial_{G \circ K_1}(Y) \). From Lemma 2.12, \( \partial_{G \circ K_1}(A) = \partial(G \circ K_1) \), where \( A = Y \cap V(G) \). Now, from Lemma 2.10, \( B_{G \circ K_1}(A) = B_G(A) \cup S \), where \( S = \bigcup_{v \in A} V(K^v_1) \), and \( \partial_{G \circ K_1}(A) = |B_G(A)| \).
Thus, \( \partial(G \circ K_1) \leq \Psi(G) \). Therefore, \( \partial(G \circ K_1) = \Psi(G) \). \( \square \)

**Lemma 2.14.** Let \( G \) and \( H \) be connected graphs with \( |V(H)| \geq 3 \). If \( Y \subseteq V(G \circ H) \) and \( \partial_{G \circ H}(Y) = \partial(G \circ H) \), then \( Y \cap V(v + H^v) \neq \emptyset \) for each \( v \in V(G) \). Moreover, \( \partial_{G \circ H}(V(G)) = \partial(G \circ H) \).
Proof: Suppose there exists \( v \in V(G) \) such that \( Y \cap V(v + H^v) = \emptyset \). Pick \( u \in V(H^v) \) with maximum degree in \( H^v \) and set \( X = Y \cup \{u\} \). Since \( |V(H^v)| \geq 3 \), \( H^v \) is connected and \( u \) is of maximum degree, \( |N_{Go^H}(u) \setminus X| \geq 3 \). Then
\[
B_{Go^H}(X) = \left[ \cup_{y \in Y}(N_{Go^H}(y) \setminus X) \right] \cup (N_{Go^H}(u) \setminus X).
\]
If \( v \in \cup_{y \in Y}(N_{Go^H}(y) \setminus X) \), then
\[
[\cup_{y \in Y}(N_{Go^H}(y) \setminus X)] \cap [(N_{Go^H}(u) \setminus X) \{v\}] = \emptyset.
\]
Therefore,
\[
\partial_{Go^H}(X) \geq |\cup_{y \in Y}(N_{Go^H}(y) \setminus X)| + 2 - |X| = |\cup_{y \in Y}(N_{Go^H}(y) \setminus X)| - |Y| + 1.
\]
Since \( N_{Go^H}(y) \setminus X = N_{Go^H}(y) \setminus Y \) for all \( y \in Y \), it follows that
\[
\partial_{Go^H}(X) > \partial_{Go^H}(Y) + 1 > \partial_{Go^H}(Y) = \partial(G \circ H),
\]
contrary to our assumption. Therefore, \( Y \cap V(v + H^v) \neq \emptyset \) for each \( v \in V(G) \).
Now, let \( X = V(G) \). Then
\[
B_{Go^H}(X) = \cup_{v \in V(G)} V(H^v).
\]
Moreover, we have
\[
B_{Go^H}(Y) = \cup_{y \in Y}(N_{Go^H}(y) \setminus Y) \\
= \cup_{y \in Y \setminus X}(N_{Go^H}(y) \setminus Y) \cup (\cup_{y \in Y \cap X}(N_{Go^H}(y) \setminus Y)).
\]
Let \( y \in Y \setminus X \). Then, there exists a \( v \in V(G) \) such that \( y \in V(v + H^v) \).
By Lemma 2.1, \( N_{Go^H}(y) \setminus Y \subseteq (V(H^v) \{y\}) \cup \{v\} \). Thus,
\[
|N_{Go^H}(y) \setminus Y| \leq |V(H^v)| = |V(H)|.
\]
Let \( y \in Y \cap X \) and \( R_y = \{u \in V(G) \setminus Y : uy \in E(G)\} \). Then
\[
N_{Go^H}(y) \setminus Y = V(H^y) \cup R_y.
\]
Since \( R_y \subseteq \cup_{z \in Y \setminus X}(N_{Go^H}(z) \setminus Y) \neq \emptyset \), it follows that
\[
|B_{Go^H}(Y)| \leq |\cup_{v \in V(G)} V(H^v)| = |B_{Go^H}(X)|.
\]
Also, since \( Y \cap V(v + H^v) \neq \emptyset \) for all \( v \in V(G) \), \( |X| = |V(G)| \leq |Y| \). Thus,
\[
\partial_{Go^H}(Y) = |B_{Go^H}(Y)| - |Y| \\
\leq |B_{Go^H}(X)| - |X| \\
= \partial_{Go^H}(X) \\
= \partial_{Go^H}(V(G)).
\]
Therefore, \( \partial(G \circ H) = \partial_{Go^H}(V(G)) \). \( \square \)
Lemma 2.15. Let \( G \) be a connected graph and \( Y \subseteq V(G \circ K_2) \) such that \( \partial_{G \circ K_2}(Y) = \partial(G \circ K_2) \). If \( Y \cap V(v + K_2^v) = \emptyset \) for some \( v \in V(G) \), then there exists \( x \in Y \) such that \( v \in N_{G \circ K_2}(x) \) and \( \partial_{G \circ K_2}(Y \cup \{z\}) = \partial(G \circ K_2) \), where \( z \in V(v + K_2^v) \). In particular, \( \partial(G \circ K_2) = \partial_{G \circ K_2}(V(G)) \).

Proof: Suppose \( Y \cap V(v + K_2^v) = \emptyset \) for some \( v \in V(G) \). Suppose further that \( v \notin N_{G \circ K_2}(x) \) for all \( x \in Y \). Let \( X = Y \cup \{v\} \). Since

\[
|N_{G \circ K_2}(v) \backslash X| = |N_{G \circ K_2}(v) \backslash Y| \geq 2
\]

and

\[
B_{G \circ K_2}(X) = [\bigcup_{u \in Y} (N_{G \circ K_2}(u) \backslash X)] \cup (N_{G \circ K_2}(v) \backslash X)
\]

\[
= [\bigcup_{u \in Y} (N_{G \circ K_2}(u) \backslash Y)] \cup (N_{G \circ K_2}(v) \backslash Y),
\]

it follows that

\[
\partial_{G \circ K_2}(X) = |B_{G \circ K_2}(Y)| + |N_{G \circ K_2}(v) \backslash Y| - |X|
\]

\[
= |B_{G \circ K_2}(Y)| - |Y| + |N_{G \circ K_2}(v) \backslash Y| - 1
\]

\[
\geq \partial_{G \circ K_2}(Y) + 1
\]

\[
> \partial_{G \circ K_2}(Y),
\]

contrary to our assumption of \( Y \). Thus there exists \( x \in Y \) such that \( v \in N_{G \circ K_2}(x) \). Next, let \( z \in V(v + K_2^v) \), where \( Y \cap V(v + K_2^v) = \emptyset \). Consider the following cases.

Case 1. Suppose \( z \neq v \). Then \( v \in N_{G \circ K_2}(z) \) and \( |N_{G \circ K_2}(z) \backslash (Y \cup \{z\})| = 2 \). Since \( v \notin N_{G \circ K_2}(u) \) for every \( u \in Y \), it follows that

\[
B_{G \circ K_2}(Y \cup \{z\}) = [\bigcup_{u \in Y} (N_{G \circ K_2}(u) \backslash Y)] \cup (N_{G \circ K_2}(z) \backslash Y \cup \{z\})
\]

\[
= B_{G \circ K_2}(Y) \cup (N_{G \circ K_2}(z) \backslash Y \cup \{z\}).
\]

As shown earlier, \( v \) must be in \( B_{G \circ K_2}(Y) \). Thus,

\[
|B_{G \circ K_2}(Y \cup \{z\})| = |B_{G \circ K_2}(Y)| + |N_{G \circ K_2}(z) \backslash (Y \cup \{z\})| - 1
\]

\[
= |B_{G \circ K_2}(Y)| + 1.
\]

Consequently,

\[
\partial_{G \circ K_2}(Y \cup \{z\}) = \partial_{G \circ K_2}(Y) = \partial(G \circ K_2) = \partial_{G \circ K_2}(Y).
\]

Case 2. Suppose \( z = v \). Then

\[
B_{G \circ K_2}(Y \cup \{v\}) = [\bigcup_{u \in Y} (N_{G \circ K_2}(u) \backslash Y \cup \{v\})] \cup (N_{G \circ K_2}(v) \backslash (Y \cup \{v\}).
\]
Since $v \in B_{G \circ K_2}(Y)$ and $\bigcup_{u \in Y}[N_{G \circ K_2}(u) \setminus (Y \cup \{v\})] = B_{G \circ K_2}(Y) \setminus \{v\}$, it follows that $|\bigcup_{u \in Y}[N_{G \circ K_2}(u) \setminus (Y \cup \{v\})]| = |B_{G \circ K_2}(Y)| - 1$. Also, since $|N_{G \circ K_2}(v) \setminus (Y \cup \{v\})| = 2$, we have
\[
|B_{G \circ K_2}(Y \cup \{v\})| = |B_{G \circ K_2}(Y)| - 1 + |N_{G \circ K_2}(v) \setminus (Y \cup \{v\})| = |B_{G \circ K_2}(Y)| + 1.
\]
Therefore, $\partial_{G \circ K_2}(Y \cup \{v\}) = \partial_{G \circ K_2}(Y) = \partial(G \circ K_2)$.

We may assume then that $Y \cap V(v + K_2^v) \neq \emptyset$, for all $v \in V(G)$. Clearly, $|Y \cap V(v + K_2^v)| = 1$ for all $v \in V(G)$. Now, let $y \in Y \setminus V(G)$ and let $w \in V(G)$ such that $y \in V(w + K_2^w)$. Set $Y^* = (Y \setminus \{y\}) \cup \{w\}$.

Claim: $\partial_{G \circ H}(Y) = \partial_{G \circ H}(Y^*)$

Let $u \in V(G \circ H) \setminus Y$. Then there exists $x \in V(G)$ such that $u \in V(x + K_2^x)$. Let $v_x \in Y \cap V(x + K_2^x)$. Since $u \notin Y$, $u \notin (N_{G \circ H}(v_x) \setminus Y)$. Therefore, $u \in B_{G \circ H}(Y)$. Consequently, $B_{G \circ H}(Y) = V(G \circ H) \setminus Y$.

Similarly, $B_{G \circ H}(Y^*) = V(G \circ H) \setminus Y^*$. Since $|Y| = |Y^*|$, it follows that $\partial_{G \circ H}(Y) = \partial_{G \circ H}(Y^*)$.

Repeating the process for every $y \in Y \setminus V(G)$, it follows that
\[
\partial(G \circ K_2) = \partial_{G \circ H}(Y) = \partial_{G \circ H}(V(G)).
\]
This proves the assertion. \(\square\)

**Theorem 2.16.** Let $G$ and $H$ be connected graphs with $n = |V(H)| \geq 2$. Then $\partial(G \circ H) = |V(G)|(|n - 1|)$.

**Proof:** Let $X = V(G)$. By Lemma 2.14 and Lemma 2.15,
\[
\partial_{G \circ H}(X) = \partial(G \circ H).
\]
Since $B_G(X) = \emptyset$, Lemma 2.10 gives the desired result. \(\square\)

The **lexicographic product** of two graphs $G$ and $H$, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ whose elements satisfy the adjacency condition: $(u_1, v_1)$ is adjacent to $(u_2, v_2)$ if and only if $u_1u_2 \in E(G)$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

Let $S \subseteq V(G[H])$. The $G$-projection $S_G$ of $S$ and the $H$-projection $S_H$ of $S$ are defined as follows:
\[
S_G = \{u : (u, v) \in S \text{ for some } v \in V(H)\},
S_H = \{v : (u, v) \in S \text{ for some } u \in V(G)\}.
\]

**Theorem 2.17** Let $G$ be a connected graph of order $n$ and $H$ a connected graph of order $m \geq 3$ with $\Delta(H) = m - 1$. Then $\partial(G[H]) = nm - 2\gamma(G)$. 
Proof: Let $S$ be a subset of $V(G[H])$ with $\partial(G[H]) = \partial_{G[H]}(S)$. Define the set $S_G = \{a \in V(G) : (a, b) \in S \text{ for some } b \in V(H)\}$ and suppose that $S_G$ is not a dominating set of $G$. Then there exists a vertex $x \in V(G) \setminus N[S_G]$. This implies that $x \notin S_G$ and $x \notin N(v)$ for all $v \in S_G$. Pick $a \in V(H)$ such that $\deg_H(a) = \Delta(H)$ and consider the set $S^* = S \cup \{(x, a)\}$. Set $K = \{(x, u) : u \in V(H) \setminus \{a\}\}$. Then $|K| \geq 2$ and $K \cap B(S) = \emptyset$. Also, $B(S) \cup K \subseteq B(S^*)$. Thus,

$$
\partial(S^*) = |B(S^*)| - |S^*|
\geq |B(S)| + |K| - |S| - 1
\geq |B(S)| + 2 - |S| - 1
\geq |B(S)| - |S| + 1
> \partial(S).
$$

This contradicts the property of $S$. Therefore, $S_G$ is a dominating set of $G$. Consequently, $B(S) = V(G[H]) \setminus S$ and $\partial(G[H]) = nm - 2|S|$. Now, suppose $S_G$ is not a minimum dominating set of $G$, say $S_1$ is a dominating set of $G$ with $|S_1| = \gamma(G)$. Choose $y \in V(H)$ with $\deg_H(y) = m - 1$, and let $S' = S_1 \times \{y\}$. Then $B(S') = V(G[H]) \setminus S_1$ and

$$
\partial(S') = |B(S')| - |S'|
= |V(G[H])| - |S_1| - |S_1|
= nm - 2\gamma(G)
> nm - 2|S_G|
\geq nm - 2|S|
= \partial(S).
$$

Again, this contradicts the property of $S$. Hence, $S_G$ is a minimum dominating set of $G$. Furthermore, since

$$
\partial(T) = nm - 2|S_G|
= nm - 2\gamma(G)
\geq nm - 2|S|
= \partial(S)
$$

where, $T = S_G \times \{b\}$ for $b \in V(H)$, it follows that $|S| = \gamma(G)$. Therefore, $\partial(G[H]) = nm - 2\gamma(G)$. \hfill \square

**Theorem 2.18** Let $G$ and $H$ be connected non-complete graphs of orders $n$ and $m \geq 4$. Suppose further that $\Delta(H) \neq m - 1$. Then $\partial(G[H]) = nm - 2\gamma_1(G)$. 


Proof: Let $S$ be a subset of $V(G[H])$ with $\partial(G[H]) = \partial(S)$. Suppose $S_G$ is not a total dominating set of $G$. Then there exists a vertex $x \in V(G)\setminus N(S_G)$. This implies that $x \notin N(v)$ for all $v \in S_G$. Consider the following cases:

Case 1. Suppose $x \notin S_G$. Pick $a \in V(H)$ of maximum degree $\Delta(H)$ and consider the set $S^* = S \cup \{(x,a)\}$. Set $K = \{(x,u) : u \in V(H)\setminus\{a\}\}$. Then $|K| \geq 2$ and $K \cap B(S) = \emptyset$. Also, $B(S^*) = B(S) \cup K$. Thus

$$\partial(S^*) = |B(S)| + |K| - |S| - 1 \geq |B(S)| - |S| + 1 > \partial(S).$$

This contradicts the property of $S$.

Case 2. Suppose $x \in S_G$. Let $K_x = \{(x,u) : u \in V(H)\}$ and let $w \in V(H)$ with $\deg_H(w) = \Delta(H)$. Pick $y \in V(G)$ with $x \in N(y)$ and $b \in V(H)$. Consider $S^* = T \cup \{(x,w)\} \cup \{(y,b)\}$, where $T = S \setminus K_x$. Notice that $K_x \subseteq N((y,b))$. Hence $[B(S) \cup N((y,b))] \subseteq B(S^*)$ and

$$\partial(S^*) \geq |B(S)| + |N((y,b))\setminus B(S)| - |T| - 2.$$ 

Since $x \in S_G$, there exists $(x,q) \in S$. Hence $(x,q) \in K_x \setminus B(S)$. Further, since $\Delta(H) < m-1$, there exists $p \in V(H)$ such that $qp \notin E(H)$; hence $(x,q)(x,p) \notin E(G[H])$. If $(x,p) \notin B(S)$, then $(x,p) \in K_x \setminus B(S)$. If $(x,p) \in B(S)$, then there exists $(x,t) \in S$ such that $(x,p)(x,t) \in E(G[H])$. Of course, $q \neq t$; hence $(x,q) \neq (x,t)$ and $(x,t) \in K_x \setminus B(S)$. In either case, $|N((y,b))\setminus B(S)| \geq 2$. Therefore, since $|T| < |S|$, $\partial(S^*) > \partial(S)$ contradicting our assumption of the set $S$. Therefore $S_G$ is a total dominating set of $G$. Consequently, $B(S) = V(G[H]) \setminus S$ and $\partial(G[H]) = nm - 2|S|$. Now, suppose $S_G$ is not a minimum total dominating set of $G$, say $S_1$ is a total dominating set of $G$ with $|S_1| = \gamma_t(G)$. Choose $y \in V(H)$ and let $S' = S_1 \times \{y\}$. Then $B(S') = V(G[H]) \setminus S_1$ and

$$\partial(S') = nm - 2\gamma_t(G) > nm - 2|S_G| \geq nm - 2|S| = \partial(S).$$

Again, this contradicts the property of $S$. Hence $S_G$ must be a minimum total dominating set of $G$. Furthermore, since $T = S_G \times \{b\}$ for $b \in V(H)$, it follows that $|S| = \gamma_t(G)$. Therefore, $\partial(G[K_m]) = nm - 2\gamma_t(G)$. □

References


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