Additive and Quadratic Functional in Equalities in Non-Archimedean Normed Spaces

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Abstract
In this paper, we solve the additive functional inequality
\[ \|f(x + y) - f(x) - f(y)\| \leq \left\| f\left(\frac{x + y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right\| \quad (1) \]
and the quadratic functional inequality
\[ \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \left\| f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right\| \quad (2) \]
in normed spaces.
Moreover, we prove the Hyers-Ulam stability of the functional inequalities (1) and (2) in Banach spaces.

Furthermore, we investigate the additive functional inequality

$$\|f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\| \leq \|f(x+y) - f(x) - f(y)\|$$

(3)

and the quadratic functional inequality

$$\|f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\|$$

$$\leq \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|$$

(4)

in non-Archimedean normed spaces.

Moreover, we prove the Hyers-Ulam stability of the functional inequalities (3) and (4) in non-Archimedean Banach spaces.

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1 Introduction and preliminaries

A valuation is a function \(| \cdot |\) from a field \(K\) into \([0, \infty)\) such that 0 is the unique element having the 0 valuation, \(|rs| = |r| \cdot |s|\) and the triangle inequality holds, i.e.,

$$|r + s| \leq |r| + |s|, \quad \forall r, s \in K.$$  

A field \(K\) is called a valued field if \(K\) carries a valuation. The usual absolute values of \(\mathbb{R}\) and \(\mathbb{C}\) are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r + s| \leq \max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function \(| \cdot |\) is called a non-Archimedean valuation, and the field is called a non-Archimedean field. Clearly \(|1| = |1| = 1\) and \(|n| \leq 1\) for all \(n \in \mathbb{N}\). A trivial example of a non-Archimedean valuation is the function \(| \cdot |\) taking everything except for 0 into 1 and \(|0| = 0\).

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.
Definition 1.1 ([8]) Let $X$ be a vector space over a field $K$ with a non-Archimedean valuation $|·|$. A function $∥·∥ : X → [0, ∞)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) $∥x∥ = 0$ if and only if $x = 0$;
(ii) $∥rx∥ = |r∥x∥$ ($r ∈ K, x ∈ X$);
(iii) the strong triangle inequality

$$∥x + y∥ ≤ \max\{∥x∥, ∥y∥\}, \quad ∀x, y ∈ X$$

holds. Then $(X, ∥·∥)$ is called a non-Archimedean normed space.

Definition 1.2 (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then the sequence $\{x_n\}$ is called Cauchy if for a given $ε > 0$ there is a positive integer $N$ such that

$$∥x_n - x_m∥ ≤ ε$$

for all $n, m ≥ N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then the sequence $\{x_n\}$ is called convergent if for a given $ε > 0$ there are a positive integer $N$ and an $x ∈ X$ such that

$$∥x_n - x∥ ≤ ε$$

for all $n ≥ N$. Then we call $x ∈ X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n→∞} x_n = x$.

(iii) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a non-Archimedean Banach space.


The functional equation

$$f(x + y) = f(x) + f(y)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.
The functional equation
\[ f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y) \]
is called the Jensen equation.

The functional equation
\[ f(x+y) + f(x-y) = 2f(x) + 2f(y) \]
is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The stability of quadratic functional equation was proved by Skof [12] for mappings \( f : E_1 \to E_2 \), where \( E_1 \) is a normed space and \( E_2 \) is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain \( E_1 \) is replaced by an Abelian group.

The functional equation
\[ f\left(\frac{x+y}{2}\right) + \left(\frac{x-y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y) \]
is called a Jensen type quadratic equation.

In [5], Gilányi showed that if \( f \) satisfies the functional inequality
\[ \|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \] (5)
then \( f \) satisfies the Jordan-von Neumann functional equation
\[ 2f(x) + 2f(y) = f(xy) + f(xy^{-1}). \]


In Section 2, we solve the additive functional inequality (1) and prove the Hyers-Ulam stability of the additive functional inequality (1) in Banach spaces.

In Section 3, we solve the quadratic functional inequality (2) and prove the Hyers-Ulam stability of the quadratic functional inequality (2) in Banach spaces.

In Section 4, we solve the additive functional inequality (3) and prove the Hyers-Ulam stability of the additive functional inequality (3) in non-Archimedean Banach spaces.

In Section 5, we solve the quadratic functional inequality (4) and prove the Hyers-Ulam stability of the quadratic functional inequality (4) in non-Archimedean Banach spaces.
2 Additive functional inequalities in Banach spaces

Throughout this section, assume that $X$ is a normed space with norm $\| \cdot \|$ and that $Y$ is a Banach space with norm $\| \cdot \|$.

**Lemma 2.1** A mapping $f : X \to Y$ satisfies

$$\| f(x + y) - f(x) - f(y) \| \leq \| f\left(\frac{x + y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \|$$

(6)

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

**Proof.** Assume that $f : X \to Y$ satisfies (6).

Letting $x = y = 0$ in (6), we get $\| f(0) \| \leq 0$. So $f(0) = 0$.

Letting $y = -x$ in (6), we get $\| f(x) + f(-x) \| \leq \frac{1}{2} \| f(x) + f(-x) \|$ for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $y = x$ in (6), we get $\| f(2x) - 2f(x) \| \leq 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (6) that $\| f(x+y) - f(x) - f(y) \| \leq \frac{1}{2} \| f(x+y) - f(x) - f(y) \|$ and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

The converse is obviously true.

**Corollary 2.2** A mapping $f : X \to Y$ satisfies

$$f(x + y) = f\left(\frac{x + y}{2}\right) + \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Now, we prove the Hyers-Ulam stability of the additive functional inequality (6) in Banach spaces.

**Theorem 2.3** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that

$$\| f(x + y) - f(x) - f(y) \| \leq \| f\left(\frac{x + y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \| + \theta(\|x\|^r + \|y\|^r)$$

(7)

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\| f(x) - h(x) \| \leq \frac{2\theta}{2^r - 2} \|x\|^r$$

(8)

for all $x \in X$. 
Proof. Letting $y = x$ in (7), we get
\[ \|f(2x) - 2f(x)\| \leq 2\|x\|^r \] (9)
for all $x \in X$. So $\|f(x) - 2f\left(\frac{x}{2}\right)\| \leq \frac{2}{2^2}\|x\|^r$ for all $x \in X$. Hence
\[
\left\|2^lf\left(\frac{x}{2^l}\right) - 2^mf\left(\frac{x}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1} \left\|2^lf\left(\frac{x}{2^j}\right) - 2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\|
\leq \frac{2}{2^r} \sum_{j=l}^{m-1} 2^j \|x\|^r
\] (10)
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (10) that the sequence $\{2^nf\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{2^nf\left(\frac{x}{2^n}\right)\}$ converges. So one can define the mapping $h : X \to Y$ by $h(x) := \lim_{n \to \infty} 2^nf\left(\frac{x}{2^n}\right)$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (10), we get (8).

It follows from (7) that
\[
\|h(x + y) - h(x) - h(y)\|
= \lim_{n \to \infty} 2^n \left\|f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\|
\leq \lim_{n \to \infty} 2^n \left\|f\left(\frac{x+y}{2^{n+1}}\right) - \frac{1}{2} f\left(\frac{x}{2^n}\right) - \frac{1}{2} f\left(\frac{y}{2^n}\right)\right\| + \lim_{n \to \infty} \frac{2^n \theta}{2^r} (\|x\|^r + \|y\|^r)
= \left\|h\left(\frac{x+y}{2}\right) - \frac{1}{2} h(x) - \frac{1}{2} h(y)\right\|
\] for all $x, y \in X$. So
\[
\|h(x + y) - h(x) - h(y)\| \leq \left\|h\left(\frac{x+y}{2}\right) - \frac{1}{2} h(x) - \frac{1}{2} h(y)\right\|
\]
for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (8). Then we have
\[
\|h(x) - T(x)\| = 2^n \left\|h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|
\leq 2^n \left(\left\|h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right)
\leq \frac{4 \cdot 2^n}{(2^r - 2)2^r \theta} \|x\|^r,
\]
which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (8).
Theorem 2.4 Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (7). Then there exists a unique additive mapping $h : X \to Y$ such that

$$\| f(x) - h(x) \| \leq \frac{2\theta}{2 - 2^r} \| x \|^r$$

for all $x \in X$.

Proof. It follows from (9) that $\| f(x) - \frac{1}{2}f(2x) \| \leq \theta \| x \|^r$ for all $x \in X$. Hence

$$\left\| \frac{1}{2^m} f(2^m x) - \frac{1}{2^l} f(2^l x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (12) that the sequence $\left\{ \frac{1}{2^m} f(2^m x) \right\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\left\{ \frac{1}{2^m} f(2^m x) \right\}$ converges. So one can define the mapping $h : X \to Y$ by $h(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (12), we get (11).

The rest of the proof is similar to the proof of Theorem 2.3.

3 Quadratic functional inequalities in Banach spaces

Throughout this section, assume that $X$ is a normed space with norm $\| \cdot \|$ and that $Y$ is a Banach space with norm $\| \cdot \|$.

Lemma 3.1 A mapping $f : X \to Y$ satisfies

$$\| f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \leq \left\| f \left( \frac{x + y}{2} \right) + f \left( \frac{x - y}{2} \right) - \frac{1}{2} f(x) - \frac{1}{2} f(y) \right\|$$

for all $x, y \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (13).

Letting $x = y = 0$ in (13), we get $\| 2f(0) \| \leq \| f(0) \|$. So $f(0) = 0$.

Letting $y = x$ in (13), we get $\| f(2x) - 4f(x) \| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus $f \left( \frac{x}{2} \right) = \frac{1}{4} f(x)$ for all $x \in X$. 
It follows from (13) that
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|
\leq \frac{1}{4}\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|
\]
and so \(f(x + y) + f(x - y) = 2f(x) + 2f(y)\) for all \(x, y \in X\).
The converse is obviously true.

**Corollary 3.2** A mapping \(f : X \rightarrow Y\) satisfies

\[
f(x + y) + f(x - y) = f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) + \frac{3}{2}f(x) + \frac{3}{2}f(y)
\]
for all \(x, y \in X\) if and only if \(f : X \rightarrow Y\) is quadratic.

Now, we prove the Hyers-Ulam stability of the quadratic functional inequality (13) in Banach spaces.

**Theorem 3.3** Let \(r > 2\) and \(\theta\) be nonnegative real numbers, and let \(f : X \rightarrow Y\) be a mapping such that

\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \left\| f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \right\| + \theta(||x||^r + ||y||^r)
\]
for all \(x, y \in X\). Then there exists a unique quadratic mapping \(h : X \rightarrow Y\) such that

\[
\|f(x) - h(x)\| \leq \frac{2\theta}{2r - 4}||x||^r
\]
for all \(x \in X\).

**Proof.** Letting \(x = y = 0\) in (14), we get \(\|2f(0)\| \leq ||f(0)||\). So \(f(0) = 0\).
Letting \(y = x\) in (14), we get

\[
\|f(2x) - 4f(x)\| \leq 2\theta||x||^r
\]
for all \(x \in X\). So \(\|f(x) - 4f\left(\frac{x}{2}\right)\| \leq \frac{\theta}{2}||x||^r\) for all \(x \in X\). Hence

\[
\left\| 4^lf\left(\frac{x}{2^l}\right) - 4^mf\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\|
\leq \frac{2}{2^r} \sum_{j=l}^{m-1} \frac{4^j}{2^{rj}} \theta||x||^r
\]
for all \(x \in X\), \(l \leq m\).
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (17) that the sequence $\{4^nf(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^nf(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by $h(x) := \lim_{n \to \infty} 4^nf(\frac{x}{2^n})$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (17), we get (15).

It follows from (14) that

$$\|h(x + y) + h(x - y) - 2h(x) - 2h(y)\|$$

$$= \lim_{n \to \infty} 4^n \left\| f \left( \frac{x + y}{2^n} \right) + f \left( \frac{x - y}{2^n} \right) - 2f \left( \frac{x}{2^n} \right) - 2f \left( \frac{y}{2^n} \right) \right\|$$

$$\leq \lim_{n \to \infty} 4^n \left\| f \left( \frac{x + y}{2^{n+1}} \right) + f \left( \frac{x - y}{2^{n+1}} \right) - \frac{1}{2} f \left( \frac{x}{2^n} \right) - \frac{1}{2} f \left( \frac{y}{2^n} \right) \right\|$$

$$+ \lim_{n \to \infty} \frac{4^n\theta}{2^{nr}} (\|x\|^r + \|y\|^r)$$

$$= \left\| h \left( \frac{x + y}{2} \right) + h \left( \frac{x - y}{2} \right) - \frac{1}{2} h(x) - \frac{1}{2} h(y) \right\|$$

for all $x, y \in X$. So

$$\|h(x + y) + h(x - y) - 2h(x) - 2h(y)\|$$

$$\leq \left\| h \left( \frac{x + y}{2} \right) + h \left( \frac{x - y}{2} \right) - \frac{1}{2} h(x) - \frac{1}{2} h(y) \right\|$$

for all $x, y \in X$. By Lemma 3.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (15). Then we have

$$\|h(x) - T(x)\| = 4^n \left\| h \left( \frac{x}{2^n} \right) - T \left( \frac{x}{2^n} \right) \right\|$$

$$\leq 4^n \left( \left\| h \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right) \right\| + \| T \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right) \| \right)$$

$$\leq \frac{4 \cdot 4^n}{(2^r - 4)2^{nr}} \theta \|x\|^r,$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (15).

**Theorem 3.4** Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (14). Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{4 - 2^r} \|x\|^r$$

(18)

for all $x \in X$. 
Proof. It follows from (16) that \( \|f(x) - \frac{1}{4}f(2x)\| \leq \frac{\theta}{2}\|x\|^r \) for all \( x \in X \). Hence
\[
\left\| \frac{1}{4^l}f(2^lx) - \frac{1}{4^m}f(2^mx) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j}f(2^jx) - \frac{1}{4^{j+1}}f(2^{j+1}x) \right\|
\leq \sum_{j=l}^{m-1} \frac{2^j \theta}{4^{j+1}} \|x\|^r
\]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (19) that the sequence \( \{\frac{1}{4^n}f(2^nx)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{\frac{1}{4^n}f(2^nx)\} \) converges. So one can define the mapping \( h : X \to Y \) by \( h(x) := \lim_{n \to \infty} \frac{1}{4^n}f(2^nx) \) for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (19), we get (18).

The rest of the proof is similar to the proof of Theorem 3.3.

4 Additive functional inequalities in non-Archimedean Banach spaces

Throughout this section, assume that \( X \) is a non-Archimedean normed space and that \( Y \) is a non-Archimedean Banach space. Assume that \( |2| \neq 1 \).

Lemma 4.1 An odd mapping \( f : X \to Y \) satisfies
\[
\left\| f \left( \frac{x+y}{2} \right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \right\| \leq \|f(x + y) - f(x) - f(y)\| \quad (20)
\]
for all \( x, y \in X \) if and only if \( f \) is additive.

Proof. Letting \( y = 0 \) in (20), we get \( \|f \left( \frac{x}{2} \right) - \frac{1}{2}f(x)\| \leq 0 \) and so \( f \left( \frac{x}{2} \right) = \frac{1}{2}f(x) \) for all \( x \in X \). Thus
\[
\frac{1}{|2|} \|f(x) - f(x) - f(y)\| = \left\| \frac{1}{2}(f(x + y) - f(x) - f(y)) \right\|
= \left\| f \left( \frac{x+y}{2} \right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \right\|
\leq \|f(x + y) - f(x) - f(y)\|
\]
for all \( x, y \in X \). Since \( |2| < 1 \), \( f(x + y) = f(x) + f(y) \) for all \( x, y \in X \).

The converse is obviously true.

Corollary 4.2 An odd mapping \( f : X \to Y \) satisfies
\[
f \left( \frac{x+y}{2} \right) + \frac{1}{2}f(x) + \frac{1}{2}f(y) = f(x + y)
\]
for all \( x, y \in X \) if and only if \( f : X \to Y \) is additive.
We prove the Hyers-Ulam stability of the additive functional inequality (20) in non-Archimedean Banach spaces.

**Theorem 4.3** Let \( r < 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be an odd mapping such that
\[
\| f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(x) - \frac{1}{2} f(y) \| \leq \| f(x) - f(y) \| + \theta (\|x\|^r + \|y\|^r) \tag{21}
\]
for all \( x, y \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that
\[
\| f(x) - A(x) \| \leq |2\theta| |x|^r \tag{22}
\]
for all \( x \in X \).

**Proof.** Letting \( y = 0 \) in (21), we get
\[
\| f \left( \frac{x}{2} \right) - \frac{1}{2} f(x) \| \leq \theta \|x\|^r \tag{23}
\]
for all \( x \in X \). So \( \| f(x) - 2f \left( \frac{x}{2} \right) \| \leq |2\theta| |x|^r \) for all \( x \in X \). Hence
\[
\| 2^l f \left( \frac{x}{2^l} \right) - 2^m f \left( \frac{x}{2^m} \right) \| \leq \max \left\{ \| 2^l f \left( \frac{x}{2^l} \right) - 2^{l+1} f \left( \frac{x}{2^{l+1}} \right) \|, \cdots, \| 2^{m-1} f \left( \frac{x}{2^{m-1}} \right) - 2^m f \left( \frac{x}{2^m} \right) \| \right\} \tag{24}
\]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (24) that the sequence \( \{2^k f \left( \frac{x}{2^k} \right) \} \) is Cauchy for all \( x \in X \). Since \( Y \) is a non-Archimedean Banach space, the sequence \( \{2^k f \left( \frac{x}{2^k} \right) \} \) converges. So one can define the mapping \( A : X \to Y \) by \( A(x) := \lim_{k \to \infty} 2^k f \left( \frac{x}{2^k} \right) \) for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (24), we get (22).

Now, let \( T : X \to Y \) be another additive mapping satisfying (22). Then we have
\[
\| A(x) - T(x) \| = \| 2^m A \left( \frac{x}{2^m} \right) - 2^q T \left( \frac{x}{2^q} \right) \|
\leq \max \left\{ \| 2^m A \left( \frac{x}{2^m} \right) - 2^{m+1} f \left( \frac{x}{2^{m+1}} \right) \|, \cdots, \| 2^{q-1} T \left( \frac{x}{2^{q-1}} \right) - 2^q f \left( \frac{x}{2^q} \right) \| \right\}
\leq \frac{|2|}{|2| (r-1) \theta \|x\|^r},
\]
which tends to zero as \( q \to \infty \) for all \( x \in X \). So we can conclude that \( A(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( A \).

It follows from (21) that

\[
\left\| A \left( \frac{x + y}{2} \right) - \frac{1}{2} A(x) - \frac{1}{2} A(y) \right\| \\
= \lim_{n \to \infty} 2^n \left( f \left( \frac{x + y}{2^{n+1}} \right) - \frac{1}{2} f \left( \frac{x}{2^n} \right) - \frac{1}{2} f \left( \frac{y}{2^n} \right) \right) \\
\leq \lim_{n \to \infty} 2^n \left( f \left( \frac{x + y}{2^n} \right) - f \left( \frac{x}{2^n} \right) - f \left( \frac{y}{2^n} \right) \right) \\
= \|A(x + y) - A(x) - A(y)\|
\]

for all \( x, y \in X \). So

\[
\left\| A \left( \frac{x + y}{2} \right) - \frac{1}{2} A(x) - \frac{1}{2} A(y) \right\| \leq \|A(x + y) - A(x) - A(y)\|
\]

for all \( x, y \in X \). By Lemma 4.1, the mapping \( A : X \to Y \) is additive.

**Theorem 4.4** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be an odd mapping satisfying (21). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\|f(x) - A(x)\| \leq |2|^r \theta \|x\|^r
\]

for all \( x \in X \).

**Proof.** It follows from (23) that \( \|f(x) - \frac{1}{2} f(2x)\| \leq |2|^r \theta \|x\|^r \) for all \( x \in X \).

The rest of the proof is similar to the proof of Theorem 4.3.

## 5 Quadratic functional inequalities in non-Archimedean Banach spaces

Throughout this section, assume that \( X \) is a non-Archimedean normed space and that \( Y \) is a non-Archimedean Banach space. Assume that \( |2| \neq 1 \).

**Lemma 5.1** An even mapping \( f : X \to Y \) satisfies

\[
\left\| f \left( \frac{x + y}{2} \right) + f \left( \frac{x - y}{2} \right) - \frac{1}{2} f(x) - \frac{1}{2} f(y) \right\| \\
\leq \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \tag{25}
\]

for all \( x, y \in X \) if and only if \( f \) is quadratic.
Proof. Letting \( x = y = 0 \) in (25), we get \( \|f(0)\| \leq \|2f(0)\| = |2|\|f(0)\| \). So \( f(0) = 0 \).

Letting \( y = 0 \) in (25), we get \( \|2f\left(\frac{x}{2}\right) - \frac{1}{2}f(x)\| \leq 0 \) and so \( f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \) for all \( x \in X \). Thus

\[
\frac{1}{|2|^2} \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \\
= \left\| \frac{1}{4}(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \right\| \\
= \left\| f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \right\| \\
\leq \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|
\]

for all \( x, y \in X \). Since \( |2| < 1 \), \( f(x + y) + f(x - y) = 2f(x) + 2f(y) \) for all \( x, y \in X \).

The converse is obviously true.

**Corollary 5.2** An even mapping \( f : X \to Y \) satisfies

\[
f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) + 3 \frac{1}{2}f(x) + 3 \frac{1}{2}f(y) = f(x + y) + f(x - y)
\]

for all \( x, y \in X \) if and only if \( f : X \to Y \) is quadratic.

Now, we prove the Hyers-Ulam stability of the quadratic functional inequality (25) in non-Archimedean Banach spaces.

**Theorem 5.3** Let \( r < 2 \) and \( \theta \) be nonnegative real numbers, and let \( f : X \to Y \) be an even mapping such that

\[
\left\| f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \right\| \\
\leq \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| + \theta(\|x\|^r + \|y\|^r)
\]

for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\|f(x) - Q(x)\| \leq |2|\theta\|x\|^r
\]

for all \( x \in X \).

**Proof.** Letting \( x = y = 0 \) in (26), we get \( \|f(0)\| \leq \|2f(0)\| = |2|\|f(0)\| \). So \( f(0) = 0 \).

Letting \( y = 0 \) in (26), we get

\[
\left\| 2f\left(\frac{x}{2}\right) - \frac{1}{2}f(x) \right\| \leq \theta\|x\|^r
\]

(28)
for all $x \in X$. So $\|f(x) - 4f(x/2)\| \leq |2\theta\|x\|^r$ for all $x \in X$. Hence

$$\|4^lf(x/2) - 4^m f(x/2^m)\| \leq \max\left\{\|4^lf(x/2) - 4^{l+1} f(x/2^{l+1})\|, \ldots, \|4^{m-1} f(x/2^{m-1}) - 4^m f(x/2^m)\|\right\}$$

$$\leq \max\left\{|4|^l\|f(x/2) - 4f(x/2^{l+1})\|, \ldots, |4|^{m-1}\|f(x/2^{m-1}) - 4f(x/2^m)\|\right\}$$

$$\leq \max\left\{|\frac{|4|^l}{2^{l+1}}, \ldots, \frac{|4|^{m-1}}{2^{r(m-1)}}\right\}|2\theta\|x\|^r = \frac{|2|}{|2|^{r-2}}\theta\|x\|^r$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (29) that the sequence $\{4^k f(x/2^k)\}$ is Cauchy for all $x \in X$. Since $Y$ is a non-Archimedean Banach space, the sequence $\{4^k f(x/2^k)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by $Q(x) := \lim_{k \rightarrow \infty} 4^k f(x/2^k)$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (29), we get (27).

The rest of the proof is similar to the proof of Theorem 4.3.

**Theorem 5.4** Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (26). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq |2|^{r-1}\theta\|x\|^r$$

for all $x \in X$.

**Proof.** It follows from (28) that $\|f(x) - \frac{1}{7} f(2x)\| \leq |2|^{r-1}\theta\|x\|^r$ for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 4.3 and 5.3.

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**References**


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