Structure of the Zeros
of the Twisted $q$-Tangent Polynomials

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Abstract

In [3], we introduced the twisted $q$-tangent numbers $T_{n,q,w}(x)$ and polynomials $T_{n,q,w}(x)$. In this paper, using computer, we observe the structure of complex roots of the twisted $q$-tangent polynomials $T_{n,q,w}(x)$. Finally, we give a table for the solutions of the twisted $q$-tangent polynomials $T_{n,q,w}(x)$.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: tangent numbers and polynomials, twisted $q$-tangent numbers and polynomials, complex roots

1 Introduction

In [1], we introduce the tangent numbers $T_n$ and polynomials $T_n(x)$. The tangent numbers $T_n$ are defined by the generating function:

$$\frac{2}{e^{2t} + 1} = \sum_{n=0}^{\infty} \frac{T_n t^n}{n!}.$$  \hspace{1cm} (1.1)

We introduce the tangent polynomials $T_n(x)$ as follows:

$$\left(\frac{2}{e^{2t} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!}.$$  \hspace{1cm} (1.2)
In [3], we introduced the twisted \( q \)-tangent numbers \( T_{n,q,w}(x) \) and polynomials \( T_{n,q,w}(x) \). By using these numbers and polynomials, we investigated some interesting properties. In order to study the twisted \( q \)-tangent numbers \( T_{n,q,w} \) and polynomials \( T_{n,q,w}(x) \), we must understand the structure of the twisted \( q \)-tangent numbers \( T_{n,q,w} \) and polynomials \( T_{n,q,w}(x) \). Therefore, by using computer, a realistic study for the twisted \( q \)-tangent numbers \( T_{n,q,w}(x) \) and polynomials \( T_{n,q,w}(x) \) is very interesting. It is the aim of this paper to observe an interesting phenomenon of ‘scattering’ of the zeros of the twisted \( q \)-tangent polynomials \( T_{n,q,w}(x) \) in complex plane.

2 The twisted \( q \)-tangent polynomials

Throughout this paper, we always make use of the following notations: \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{N}_0 \) denotes the set of nonnegative integers, \( \mathbb{Z} \) denotes the set of integers, \( \mathbb{R} \) denotes the set of real numbers, and \( \mathbb{C} \) denotes the set of complex numbers.

In this section, we introduce the twisted \( q \)-tangent numbers \( T_{n,q,w}(x) \) and polynomials \( T_{n,q,w}(x) \) and investigate their properties. Let \( q \) be a complex number with \( |q| < 1 \) and \( w \) be the \( p \mathbb{N} \)-th root of unity. By the meaning of (1.1) and (1.2), let us define the twisted \( q \)-tangent numbers \( T_{n,q,w} \) and polynomials \( T_{n,q,w}(x) \) as follows:

\[
F_q(t) = \frac{2}{wqe^{2t} + 1} = \sum_{n=0}^{\infty} T_{n,q,w} \frac{t^n}{n!},
\]

(2.1)

\[
F_q(x, t) = \left( \frac{2}{wqe^{2t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n,q,w}(x) \frac{t^n}{n!}.
\]

(2.2)

Observe that if \( q \to 1 \) and \( w = 1 \), then \( T_{n,q,w}(x) = T_n(x) \) and \( T_{n,q,w} = T_n \) (see [1]). By using computer, the twisted \( q \)-tangent numbers \( T_{n,q,w} \) can be determined explicitly. A few of them are

\[
T_{0,q,w} = \frac{2}{1 + wq}, \quad T_{1,q,w} = -\frac{4wq}{(1 + wq)^2}, \quad T_{2,q,w} = \frac{16w^2q^2}{(1 + wq)^3} - \frac{8wq}{(1 + wq)^2},
\]

\[
T_{3,q,w} = -\frac{96w^3q^3}{(1 + wq)^4} + \frac{96w^2q^2}{(1 + wq)^3} - \frac{16wq}{(1 + wq)^2}.
\]

The following elementary properties of twisted \( q \)-tangent polynomials \( T_{n,q,w}(x) \) are readily derived from (2.1) and (2.2). Therefore we choose to omit the details involved. More studies and results in this subject we may see references [1]-[3].
Theorem 2.1 For any positive integer $n$, we have
$$T_{n,q,w}(x) = (-1)^n w^{-1} q^{-1} T_{n,q^{-1},w^{-1}}(2-x).$$

Theorem 2.2 For any positive integer $m (=\text{odd})$, we have
$$T_{n,q,w}(x) = m^n \sum_{i=0}^{m-1} (-1)^i w^i q^i T_{n,q^m,w^m} \left( \frac{2i + x}{m} \right), \quad n \in \mathbb{N}_0.$$

Theorem 2.3 For $n \in \mathbb{N}_0$, we have
$$T_{n,q,w}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q,w} x^{n-l}.$$

By Theorem 2.3, after some elementary calculations, we have
$$\int_a^b T_{n,q,w}(x) dx = \sum_{l=0}^{n} \binom{n}{l} T_{l,q,w} \int_a^b x^{n-l} dx$$
$$= \sum_{l=0}^{n} \binom{n}{l} T_{l,q,w} \frac{x^{n-l+1}}{n-l+1} \bigg|_a^b$$
$$= \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} T_{l,q,w} x^{n-l+1} \bigg|_a^b.$$

By Theorem 2.3, we get
$$\int_a^b T_{n,q,w}(x) dx = \frac{T_{n+1,q,w}(b) - T_{n+1,q,w}(a)}{n+1}. \quad (2.3)$$

Since $T_{n,q,w}(0) = T_{n,q,w}$, by (2.3), we have the following theorem.

Theorem 2.4 For $n \in \mathbb{N}$, we have
$$T_{n,q,w}(x) = T_{n,q,w} + n \int_0^x T_{n-1,q,w}(t) dt.$$

Then, it is easy to deduce that $T_{n,q,w}(x)$ are polynomials of degree $n$. Here is the list of the first twisted $q$-tangent’s polynomials.

$$T_{0,q,w}(x) = \frac{2}{1+wq},$$
$$T_{1,q,w}(x) = \frac{-4wq}{(1+wq)^2} + \frac{2x}{1+wq},$$
$$T_{2,q,w}(x) = \frac{16q^2 w^2}{(1+qw)^3} - \frac{8qw}{(1+qw)^2} - \frac{8qw x}{(1+qw)^2} + \frac{2x^2}{1+qw},$$
$$T_{3,q,w}(x) = -\frac{96q^3 w^3}{(1+qw)^4} + \frac{96q^2 w^2}{(1+qw)^3} - \frac{16qw}{(1+qw)^2}$$
$$+ \frac{48q^2 w^2 x}{(1+qw)^3} - \frac{24qw x}{(1+qw)^2} - \frac{12qw x^2}{(1+qw)^2} + \frac{2x^3}{1+qw}.$$
3 Location of zeros of twisted $q$-tangent polynomials

In this section, we investigate the location of the zeros of the twisted $q$-tangent polynomials $T_{n,q,w}(x)$. Let $w = e^{2\pi i}$ in $\mathbb{C}$. By using a computer, we investigate the beautiful zeros of the $T_{n,q,w}(x)$. We plot the zeros of the twisted $q$-tangent polynomials $T_{n,q,w}(x)$ for $n = 30$, $q = 1/5$ and $x \in \mathbb{C}$ (Figure 1). In Figure 1(top-left), we choose $w = e^{2\pi i}$. In Figure 1(top-right), we choose $w = e^{\pi i}$. In Figure 1(bottom-left), we choose $w = e^{\frac{2\pi}{3}}$. In Figure 1(bottom-right), we choose $w = e^{\frac{\pi}{2}}$. Stacks of zeros of $T_{n,q,w}(x)$ for $1 \leq n \leq 30$, $q = 1/5$, $w = e^{\frac{\pi}{2}}$ from a 3-D structure are presented (Figure 2). Our numerical results for approximate solutions of real zeros of $T_{n,q,w}(x)$ are displayed (Tables 1, 2).
Zeros of the twisted $q$-tangent Polynomials

Figure 2: Stacks of zeros of $T_{n,q,w}(x)$ for $1 \leq n \leq 30$

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$q = 1/5, w = e^{2\pi i}$</th>
<th>$q = 1/5, w = e^{\pi i}$</th>
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</thead>
<tbody>
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<td>real zeros</td>
<td>complex zeros</td>
<td>real zeros</td>
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<td>1</td>
<td>1</td>
</tr>
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<tr>
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<td>3</td>
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<tr>
<td>8</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

We observe a remarkably regular structure of the complex roots of the twisted $q$-tangent polynomials $T_{n,q,w}(x)$. We hope to verify a remarkably regular structure of the complex roots of the twisted $q$-tangent polynomials $T_{n,q,w}(x)$ (Table 1). In Figures 1-3, $T_{n,q,w}(x)$ has not $Re(x) = 0$ and $Im(x) = 0$ reflection symmetry. Plot of real zeros of $T_{n,q,w}(x)$ for $1 \leq n \leq 30, q = 1/5$ structure is presented (Figure 3). Next, we calculated an approximate solution satisfying
Figure 3: Real zeros of $T_{n,q,w}(x)$ for $w = e^{\pi i}$ and $1 \leq n \leq 30$

$T_{n,q,w}(x) = 0$, $q = 1/5$, $w = e^{\pi i}$, and $x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $T_{n,q,w}(x) = 0$, $x \in \mathbb{R}$

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
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<td>3</td>
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<td>5</td>
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<tr>
<td>7</td>
<td>-2.833</td>
</tr>
<tr>
<td>9</td>
<td>-3.56</td>
</tr>
</tbody>
</table>

Prove or disprove: $T_{n,q,w}(x) = 0$ has $n$ distinct solutions. Find the numbers of complex zeros $C_{T_{n,q,w}(x)}$ of $T_{n,q,w}(x)$, $Im(x) \neq 0$. Since $n$ is the degree of the polynomial $T_{n,q,w}(x)$, the number of real zeros $R_{T_{n,q,w}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{T_{n,q,w}(x)} = n - C_{T_{n,q,w}(x)}$, where $C_{T_{n,q,w}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{T_{n,q,w}(x)}$ and $C_{T_{n,q,w}(x)}$.

References


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