Differentiating Total Dominating Sets in the Join, Corona and Composition of Graphs

Benjamin N. Omamalin

Bohol Island State University-Balilihan Campus
College of Technology and Allied Sciences
Magsiija, Balilihan, Bohol, Philippines

Sergio R. Canoy Jr. and Helen M. Rara

Department of Mathematics and Statistics
Mindanao State University-Iligan Institute of Technology
Tibanga Highway, Iligan City, Philippines

Abstract

Let $G = (V(G), E(G))$ be a connected graph. A subset $S$ of $V(G)$ is a total dominating set of $G$ if every vertex of $G$ is adjacent to some vertex in $S$. The set $N_G[v]$ is the set of all vertices of $G$ adjacent to $v$ including $v$. A subset $S$ of $V(G)$ is a differentiating set of $G$ if $N_G[u] \cap S \neq N_G[v] \cap S$ for every two distinct vertices $u$ and $v$ in $V(G)$. A differentiating subset $S$ of $V(G)$ which is also total dominating is called a differentiating total dominating set of $G$. The minimum cardinality of a differentiating total dominating set of $G$ is called the differentiating total domination number of $G$. In this paper we characterize the differentiating total dominating sets in the join, corona and composition of graphs.

Mathematics Subject Classification: 05C69

Keywords: domination, differentiating set, strictly differentiating set, total domination, join, corona, composition

1This research is partially funded by the Commission on Higher Education, Philippines under Faculty Development Program Phase II
1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph. The neighborhood of $v \in V(G)$ is the set $N_G(v) = \{x \in V(G) : xv \in E(G)\}$. The set $N_G[u] = N_G(u) \cup \{u\}$. The degree of $v \in V(G)$, denoted by $\deg_G(v)$, is equal to the cardinality of $N_G(v)$ and the maximum degree of $G$ is $\Delta(G) = \max \{\deg(x) : x \in V(G)\}$. A connected graph $G$ of order $n \geq 3$ is point distinguishing if for any two distinct vertices $u$ and $v$ of $G$, $N_G[u] \neq N_G[v]$. It is totally point determining if for any two distinct vertices $u$ and $v$ of $G$, $N_G(u) \neq N_G(v)$ and $N_G[u] \neq N_G[v]$. These concepts are studied in [3,7].

A subset $S$ of $V(G)$ is a total dominating set in $G$ if for every $v \in V(G)$, there exists $x \in S$ such that $xv \in E(G)$. It is a differentiating set in $G$ if $N_G[u] \cap S \neq N_G[v] \cap S$ for every two distinct vertices $u$ and $v$ of $G$. Set $S$ is said to be strictly differentiating if it is a differentiating set and $N_G[u] \cap S \neq S$ for all $u \in V(G)$. A differentiating (resp. strictly differentiating) subset $S$ of $V(G)$ which is also a total dominating set is called a differentiating total dominating (resp. strictly differentiating total dominating) set in a connected graph $G$. The minimum cardinality of a differentiating (resp. strictly differentiating) set in $G$, denoted by $dn(G)$ (resp. $sdn(G)$), is called the differentiating (resp. strictly differentiating) number of $G$. The minimum cardinality of a differentiating total dominating (resp. strictly differentiating total dominating) set in $G$, denoted by $\gamma_{DT}(G)$ (resp. $\gamma_{SDT}(G)$) is called the differentiating total domination (resp. strictly differentiating total domination) number of $G$.

Let $G$ be a connected graph and suppose that there exist (distinct) adjacent vertices $u$ and $v$ such that $N_G[u] = N_G[v]$. Then $N_G[u] \cap S = N_G[v] \cap S$ for any subset $S$ of $V(G)$. This implies that $G$ cannot have a differentiating set. Thus, unless otherwise stated, throughout this paper any graph considered is a point distinguishing graph.

The concepts of differentiating set, strictly differentiating set, differentiating total dominating set and the associated parameters are studied in [1,2,4,5,6]

Remark 1.1 Let $G$ be a connected graph of order $n \geq 3$.

Then $3 \leq \gamma_{DT}(G) \leq n$.

Remark 1.2 Let $G$ be a connected graph of order $n \geq 3$ with $\Delta(G) \leq n - 2$.

Then $dn(G) \leq \gamma_D(G) \leq \gamma_{DT}(G) \leq \gamma_{SDT}(G)$ and $dn(G) \leq sdn(G) \leq \gamma_{SDT}(G)$.

Lemma 1.3 [?] Let $G$ be connected graph of order $n \geq 3$ such that $dn(G) < \gamma_D(G)$. Then $1 + dn(G) = \gamma_D(G)$
2 Differentiating Total Domination in the Join of Graphs

Let $A$ and $B$ be sets which are not necessarily disjoint. The \textit{disjoint union} of $A$ and $B$, denoted by $A \cup B$, is the set obtained by taking the union of $A$ and $B$ treating each element in $A$ as distinct from each element in $B$. The \textit{join} of two graphs $G$ and $H$ is the graph $G + H$ with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

**Theorem 2.1** Let $G$ and $H$ be point distinguishing graphs of orders $m \geq 3$ and $n \geq 3$, respectively, with $\Delta(G) \leq m - 2$ and $\Delta(H) \leq n - 2$. Then $S \subseteq V(G + H)$ is a differentiating total dominating set in $G + H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are differentiating sets in $G$ and $H$, respectively, where $S_1$ or $S_2$ is strictly differentiating.

**Proof:** Let $S \subseteq V(G + H)$ be a differentiating total dominating set in $G + H$. Let $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$. Suppose $S_1 = \emptyset$. Pick distinct vertices $u$ and $v$ of $G$. Then $N_{G+H}[u] \cap S = S = N_{G+H}[v] \cap S$, contrary to our assumption of $S$. Thus, $S_1 \neq \emptyset$. Similarly, $S_2 \neq \emptyset$. Suppose now that one of $S_1$ and $S_2$ is not a differentiating set, say $S_1$ is not a differentiating set in $G$. Then there exist distinct vertices $a, b \in V(G)$ such that $N_G[a] \cap S_1 = N_G[b] \cap S_1$. Since $S_2 \subseteq N_{G+H}[a]$ and $S_2 \subseteq N_{G+H}[b]$, it follows that $N_{G+H}[a] \cap S = (N_G[a] \cap S_1) \cup S_2 = (N_G[b] \cap S_1) \cup S_2 = N_{G+H}[b] \cap S$. This, again, contradicts our assumption of $S$. Therefore, $S_1$ and $S_2$ are differentiating sets in $G$ and $H$, respectively.

Next, suppose that both $S_1$ and $S_2$ are not strictly differentiating sets in $G$ and $H$, respectively. Then there exist $z \in V(G)$ and $w \in V(H)$ such that $N_G[z] \cap S_1 = S_1$ and $N_H[w] \cap S_2 = S_2$. It follows that $N_{G+H}[z] \cap S = (N_G[z] \cap S_1) \cup S_2 = S_1 \cup (N_H[w] \cap S_2) = N_{G+H}[w] \cap S$, contrary to the fact that $S$ is a differentiating set in $G + H$. Thus, $S_1$ is a strictly differentiating set in $G$ or $S_2$ is a strictly differentiating set in $H$.

For the converse, suppose $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are differentiating sets in $G$ and $H$, respectively, where $S_1$ or $S_2$ is a strictly differentiating set. Let $x$ and $y$ be distinct vertices in $V(G + H)$. If $x, y \in V(G)$, then $N_G[x] \cap S_1 \neq N_G[y] \cap S_1$ since $S_1$ is a differentiating set in $G$. It follows that $N_{G+H}[x] \cap S = (N_G[x] \cap S_1) \cup S_2 \neq (N_G[y] \cap S_1) \cup S_2 = N_{G+H}[y] \cap S$. Similarly, $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$ if $x, y \in V(H)$. Suppose $x \in V(G)$ and $y \in V(H)$. Assume, without loss of generality, that $S_1$ is a strictly differentiating set in $G$. Then $S_1$ is not contained in $N_{G+H}[x]$. Since $S_1 \subseteq N_{G+H}[y]$, it follows that $N_{G+H}[x] \cap S \neq N_{G+H}[y] \cap S$. Accordingly, $S$ is a differentiating set in $G + H$. Let $v \in V(G + H)$. Then either $v \in V(G)$ or $v \in V(H)$. If $v \in V(G)$, then $vs \in E(G + H)$ for all $s \in S \cap V(H)$ and if $v \in V(H)$, then $va \in E(G + H)$.
for all \( a \in S \cap V(G) \). Hence, \( S \) is a total dominating set in \( G+H \). Therefore, \( S \) is a differentiating total dominating set in \( G+H \). \( \square \)

**Corollary 2.2** Let \( G \) and \( H \) be point distinguishing graphs of orders \( m \geq 3 \) and \( n \geq 3 \), respectively, with \( \Delta(G) \leq m-2 \) and \( \Delta(H) \leq n-2 \).

Then \( \gamma_{DT}(G+H) = \min\{sdn(H) + dn(G), sdn(G) + dn(H)\} \).

**Proof**: Let \( S \) be a minimum differentiating total dominating set in \( G+H \). Let \( S_1 = V(G) \cap S \) and \( S_2 = V(H) \cap S \). By Theorem 2.1, \( S_1 \) and \( S_2 \) are differentiating sets in \( G \) and \( H \), respectively, and \( S_1 \) or \( S_2 \) is a strictly differentiating set. Assume first that \( S_1 \) is a strictly differentiating set in \( G \). Then \( sdn(G) + dn(H) \leq |S_1| + |S_2| = S = \gamma_{DT}(G+H) \). If \( S_2 \) is a strictly differentiating set in \( H \), then \( sdn(H) + dn(G) \leq |S_1| + |S_2| = S = \gamma_{DT}(G+H) \).

Thus, \( \gamma_{DT}(G+H) \geq \min\{sdn(H) + dn(G), sdn(G) + dn(H)\} \).

Now suppose that \( sdn(G) + dn(H) \leq sdn(H) + dn(G) \). Let \( S^*_1 \) be a minimum strictly differentiating set in \( G \) and \( S^*_2 \) be a minimum differentiating set in \( H \). Then \( S^* = S^*_1 \cup S^*_2 \) is a differentiating total dominating set in \( G+H \) by Theorem 2.1. Thus, \( \gamma_{DT}(G+H) \leq |S^*| = |S^*_1| + |S^*_2| = sdn(G) + dn(H) \). This proves the desired equality. \( \square \)

**Theorem 2.3** Let \( G \) be a point distinguishing graph of order \( n \geq 3 \) and such that \( \Delta(G) \leq n-2 \). Then \( S \subseteq V(G+K_1) \) is a differentiating total dominating set in \( G+K_1 \) if and only if for \( v \in V(K_1) \) either \( S = S_1 \cup \{v\} \), where \( S_1 \) is a strictly differentiating set in \( G \), or \( v \notin S \) and \( S \) is a strictly differentiating total dominating set in \( G \).

**Proof**: Suppose \( S \) is a differentiating total dominating set in \( G+K_1 \) and \( v \in V(K_1) \). Suppose \( v \notin S \). Let \( S_1 = V(G) \cap S \). Since \( S \) is differentiating, \( S_1 \neq \emptyset \). Also, since \( N_{G+K_1}[v] \cap S = S_1 \), \( V(G) \cap S = S_1 \) must be strictly differentiating set in \( G \). Suppose now that \( v \notin S \). Then \( S \subseteq V(G) \) must be a total dominating set in \( G \). Since \( N_{G+K_1}[u] \cap S = N_G[u] \cap S \) for every \( u \in V(G) \) and \( N_{G+K_1}[v] \cap S = S \) and \( S \) is a differentiating set in \( G+K_1 \), \( N_G[u] \cap S \neq N_{G+K_1}[v] \cap S \) implying that \( N_G[u] \cap S \neq S \). Thus, \( S \) is a strictly differentiating set in \( H \). Hence, \( S \) is a strictly differentiating total dominating set in \( G \).

For the converse, assume first that \( S = S_1 \cup \{v\} \), where \( S_1 \) is a strictly differentiating set in \( G+K_1 \). Let \( x, y \in V(G+K_1) \), where \( x \neq y \). If either \( x \) or \( y \) is \( v \), say \( x = v \), then \( N_{G+K_1}[x] \cap S = S \neq N_G[y] \cap S_1 = N_{G+K_1}[y] \cap S_1 \), since \( S_1 = V(G) \cap S \) is a strictly differentiating set in \( G \). Suppose \( x, y \in V(G) \). Then since \( S_1 = V(G) \cap S \) is a differentiating set in \( G \), \( N_{G+K_1}[x] \cap S \neq N_{G+K_1}[y] \cap S \). This shows that \( S \) is a differentiating total dominating set in \( G+H \).

Finally, suppose \( v \notin S \) and \( S \) is a strictly differentiating total dominating
set in $G$. Then $S$ is a total dominating set in $G + K_1$. Let $x, y \in V(G + K_1)$. If $x, y \in V(G)$, then $N_{G+H}[v] \cap S = N_G[x] \cap S \neq N_G[y] \cap S = N_{G+H}[y] \cap S$. Suppose $x \in V(G)$ and $y = v$. Then $N_{G+H}[v] \cap S = S$. Since $S$ is a strictly differentiating set in $G$, $N_G[x] \cap S \neq S$. Thus,
\[N_{G+H}[v] \cap S \neq N_G[x] \cap S = N_{G+H}[x] \cap S.\]
This shows that $S$ is a differentiating total dominating set in $G + H$. \hfill \Box

**Corollary 2.4** Let $G$ be a connected non-trivial graph with
\[\Delta(G) \leq |V(G)| - 2.\] Then $\gamma_{DT}(G + K_1) = \min \{\gamma_{SDT}(G), \text{sdn}(G) + 1\}$.

**Corollary 2.5** Let $G$ be a connected non-trivial graph with
\[\Delta(G) \leq |V(G)| - 2.\] If every strictly differentiating set in $G$ is total dominating, then $\gamma_{DT}(G + K_1) = \text{sdn}(G) + 1$

### 3 Differentiating Total Domination in the Corona of Graphs

Let $G$ and $H$ be graphs of orders $m$ and $n$, respectively. The corona of two graphs $G$ and $H$ is the graph $G \circ H$ obtained by taking one copy of $G$ and $m$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$. For every $v \in V(G)$, denote $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding the join $(\{v\} + H^v)$.

**Theorem 3.1** Let $G$ and $H$ be non-trivial connected graphs of orders $m \geq 3$ and $n \geq 3$, respectively such that $\Delta(H) \leq n - 2$. Then $S \subseteq G \circ H$ is a differentiating total dominating set in $G \circ H$ if and only if for every $v \in V(G)$, one of the following is true:

(i) $v \in S$, $N_G(v) \cap S \neq \emptyset$, and $S_1 = S \cap V(H^v)$ is a differentiating set in $H^v$;

(ii) $v \in S$, $N_G(v) \cap S = \emptyset$, and $S_1 = S \cap V(H^v)$ is a strictly differentiating set in $H^v$;

(iii) $v \notin S$, $N_G(v) \cap S \neq \emptyset$, and $S_1 = S \cap V(H^v)$ is a differentiating total dominating set in $H^v$;

(iv) $v \notin S$, $N_G(v) \cap S = \emptyset$, and $S_1 = S \cap V(H^v)$ is a strictly differentiating total dominating set in $H^v$. 
Suppose first that \( v \) is differentiating in \( G \). Let \( v \in V(G) \), \( S = S \cap V(H^v) \), and let \( x, y \in V(H^v) \), where \( x \neq y \). Then

\[
N_{G \circ H}[x] \cap S = (N_{H^v}[x] \cap S_1) \cup (S \cap \{v\}), \\
N_{G \circ H}[y] \cap S = (N_{H^v}[y] \cap S_1) \cup (S \cap \{v\}),
\]

and \( N_{G \circ H}[v] \cap S = (N[v] \cap S_2) \cup (N_{v-H^v}[v] \cap S_1) \), where \( S_2 = S \cap V(G) \). Suppose first that \( v \in S \). If \( N_G(v) \cap S \neq \emptyset \), then, since \( S \) is differentiating, it follows that \( (N_{H^v}[x] \cap S_1) \cup \{v\} = N_{G \circ H}[x] \cap S \neq N_{G \circ H}[y] \cap S = (N_{H^v}[y] \cap S_1) \cup \{v\} \). Implying that \( N_{H^v}[x] \cap S_1 \neq N_{H^v}[y] \cap S_1 \). Thus, \( S_1 \) is differentiating in \( H^v \), that is, \((i)\) holds. Suppose \( N_G(v) \cap S = \emptyset \). Then, \( N_G[v] \cap S = \{v\} \). Since \( S \) is differentiating, \( S_1 \) is differentiating in \( H^v \). Moreover, since

\[
N_{G \circ H}[v] \cap S = \{v\} \cup N_{v-H^v}[v] \cap S_1 = \{v\} \cup S_1,
\]

\( S_1 \) must be strictly differentiating in \( H^v \). Hence \((ii)\) holds.

Next, suppose that \( v \notin S \). If \( N_G(v) \cap S \neq \emptyset \), then since \( S \) is differentiating, \( N_{H^v}[x] \cap S_1 = N_{G \circ H}[x] \cap S \neq N_{G \circ H}[y] \cap S = N_{H^v}[y] \cap S_1 \). This implies that \( S_1 \) is a differentiating set in \( H^v \). Since \( v \notin S \) and \( S \) is total dominating, \( S_1 \) is a total dominating set in \( H^v \). Therefore, \((iii)\) holds. Suppose that \( N_G(v) \cap S = \emptyset \). Since \( S \) is a differentiating total dominating set, \( v \notin S \) and \( N_{G \circ H}[v] = S_1 \), it follows that \( S_1 \) is a strictly differentiating total dominating set in \( H^v \), that is, \((iv)\) holds.

For the converse, suppose that \( S \) satisfies either \((i)\), \((ii)\), \((iii)\) or \((iv)\) for every \( v \in V(G) \). Let \( x \in V(G \circ H) \) and \( v \in V(G) \) such that \( x \in V(v + H^v) \). If \( x \neq v \) and \( v \in S \), then \( xv \in E(G \circ H) \). If \( v \notin S \), then, by \((iii)\), or \((iv)\), \( S_1 = V(H^v) \cap S \) is a total dominating set in \( H^v \). Hence, there exists \( y \in V(H^v) \cap S_1 \) such that \( xy \in E(H^v) \subseteq E(G \circ H) \). Therefore, \( S \) is a total dominating set in \( G \circ H \).

Next, let \( a, b \in V(G \circ H) \) with \( a \neq b \). Let \( u, v \in V(G) \) such that \( a \in V(u + H^v) \) and \( b \in V(v + H^v) \). Consider the following cases:

Case 1: Suppose that \( u = v \).
If \( a, b \in V(H^v) \), then \( N_{H^v}[a] \cap S_1 \neq N_{H^v}[b] \cap S_1 \) since \( S_1 \) is differentiating in \( H^v \) by \((i)\), \((ii)\), \((iii)\), and \((iv)\). Therefore,

\[
(N_{G \circ H}[a] \cap S) = (N_{H^v}[a] \cap S_1) \cup (S \cap \{v\}) \neq (N_{H^v}[b] \cap S_1) \cup (S \cap \{v\})
\]

Suppose \( a = v \) and \( b \in V(H^v) \). If \( N_G(v) \cap S \neq \emptyset \), say \( z \in N_G(v) \cap S \), then \( z \in [N_{G \circ H}[a] \cap S] \setminus [N_{G \circ H}[b] \cap S] \). Thus, \((N_{G \circ H}[a] \cap S) \neq (N_{G \circ H}[b] \cap S) \) implying that \( S \) is a differentiating set. If \( N_G(v) \cap S = \emptyset \), then \( V(H^v) \cap S \) is strictly differentiating in \( H^v \) by \((ii)\) and \((iv)\). Hence, there exists \( w \in V(H^v) \cap S \) such that \( w \notin N_{G \circ H}[b] \cap S \). Since \( w \in N_{G \circ H}[a] \cap S \), it follows that
Let \( G \) be a non-trivial connected graph, and \( H \) a point distinguishing graph of order \( n \geq 3 \) such that \( \Delta(H) \leq n - 2 \). Then

\[
|V(G)| \gamma_D(H) \leq \gamma_{DT}(G \circ H) \leq |V(G)| \gamma_{SDT}(H).
\]

Proof: Let \( S \) be a minimum differentiating total dominating set in \( G \circ H \). Then

\[
\gamma_{DT}(G \circ H) = |S| = \sum_{v \in V(G) \cap S} (1 + |V(H^v) \cap S|) + \sum_{v \in V(G) \setminus S} |V(H^v) \cap S|.
\]

From Theorem 3.1 (i) and (ii) and Lemma 1.3,

\[
(1 + |V(H^v) \cap S|) \geq 1 + dn(H) \geq \gamma_D(H)
\]

for every \( v \in V(G) \cap S \). From Theorem 3.1 (iii) and (iv) and Remark 1.2

\[
|V(H^v) \cap S| \geq \gamma_{DT}(H) \geq \gamma_D(H) \text{ for every } v \in V(G) \setminus S.
\]

Thus, \( \gamma_{DT}(G \circ H) = |S| \geq |V(G)| \gamma_D(H) \).

Now let \( S \) be a minimum strictly differentiating total dominating set in \( H \). For each \( v \in V(G) \), pick \( S_v \subseteq V(H^v) \), where \( \langle S_v \rangle \cong \langle S \rangle \). Then \( S = \bigcup_{v \in V(G)} S_v \) is a differentiating total dominating set in \( G \circ H \) by Theorem 3.1. Hence, \( \gamma_{DT}(G \circ H) \leq |S| = |V(G)| \gamma_{SDT}(H) \).

\[\square\]

4 Differentiating Total Domination in the Composition of Graphs

The composition of two graphs \( G \) and \( H \) is the graph \( G[H] \) with vertex-set \( V(G[H]) = V(G) \times V(H) \) and edge-set \( E(G[H]) \) satisfying the following conditions: \( (x, u)(y, v) \in E(G[H]) \) if and only if either \( xy \in E(G) \) or \( x = y \) and \( uv \in E(H) \). Observe that a non-empty subset \( C \) of \( V(G[H]) \) can be written as \( C = \bigcup_{x \in S} (\{x\} \times T_x) \), where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for every \( x \in S \).

Theorem 4.1 Let \( G \) and \( H \) be point distinguishing graph of order \( n \geq 3 \) with \( \Delta(H) \leq n - 2 \). Then
\[ C = \bigcup_{x \in \mathcal{S}} (\{x\} \times T_x), \]

where \( S \subseteq V(G) \) and \( T_x \subseteq V(H) \) for each \( x \in S \) is a differentiating total dominating set in \( G[H] \) if and only if

(i) \( S = V(G) \);

(ii) \( T_x \) is a differentiating set in \( H \) for every \( x \in V(G) \);

(iii) \( T_x \) or \( T_y \) is strictly differentiating in \( H \) whenever \( x \) and \( y \) are adjacent vertices of \( G \) with \( N_G[x] = N_G[y] \); and

(iv) \( T_x \) or \( T_y \) is (differentiating) dominating in \( H \) whenever \( x \) and \( y \) are distinct non-adjacent vertices of \( G \) with \( N_G(x) = N_G(y) \).

Proof: Suppose \( C \) is a differentiating total dominating set in \( G[H] \). Suppose there exists \( x \in V(G) \) \( \setminus \) \( S \). Choose \( a, b \in V(H) \), where \( a \neq b \). Then \( (x, a), (x, b) \notin C \) and \( (x, a) \neq (x, b) \). Since \( \{(x, c) : c \in V(H)\} \cap C = \emptyset \), it follows that

\[ N_{G[H]}[(x, a)] \cap C = N_{G[H]}[(x, b)] \cap C. \]

This means that \( C \) is not a differentiating set in \( G[H] \), a contradiction to the assumption. Thus, \( S = V(G) \).

Now, let \( x \in V(G) \) and suppose that \( T_x \) is not a differentiating set in \( H \). Then there exist distinct vertices \( p \) and \( q \) in \( V(H) \) such that

\[ N_H[p] \cap T_x = N_H[q] \cap T_x. \]

Let \( D_x = N_H[p] \cap T_x \). Then \( \{(x) \times D_x\} \subseteq C \).

Since \( N_{G[H]}[(x, p)] \cap C = \bigcup \{(y) \times T_y : y \in N_G[x]\} \cup \{(x) \times D_x\} = N_{G[H]}[(x, q)] \cap C \), it follows that \( C \) is not a differentiating set in \( G[H] \). This gives a contradiction to the assumption. Hence, \( T_x \) is a differentiating set in \( H \).

Let \( x \) and \( y \) be adjacent vertices in \( G \) with \( N_G[x] = N_G[y] \). Suppose that \( T_x \) and \( T_y \) are not strictly differentiating in \( H \). Then there exist \( c \in V(H) \) and \( d \in V(H) \) such that \( N_H(c) \cap T_x = T_x \) and \( N_H(d) \cap T_y = T_y \). It follows that \( \{(x) \times T_x\} \cup \{(y) \times T_y\} \subseteq N_{G[H]}[(x, c)] \cap N_{G[H]}[(y, d)] \). Since \( N_G[x] = N_G[y] \), it follows that \( N_{G[H]}[(x, c)] \cap C = N_{G[H]}[(y, d)] \cap C \), that is, \( C \) is not a differentiating set in \( G[H] \). This contradicts our assumption. Therefore \( T_x \) or \( T_y \) is a strictly differentiating set in \( H \).

Let \( x \) and \( y \) be distinct non-adjacent vertices of \( G \) with \( N_G(x) = N_G(y) \). If \( T_x = V(H) \) or \( T_y = V(H) \), then we are done. So suppose \( T_x \neq V(H) \) and \( T_y \neq V(H) \). If \( T_x \) is not a dominating set in \( H \), then
there exists \( a \in V(H) \setminus T_x \) such that \( ab \notin E(H) \) for all \( b \in T_x \). It follows that \((x,a) \notin C \) and \( N_{G[H]}((x,a)) \cap C = \bigcup \{ \{ z \} \times T_z : z \in N_G(x) \} \). Let \( c \in V(H) \setminus T_y \). Then \((y,c) \notin C \). Since \( N_G(x) = N_G(y) \), it follows that \( \bigcup \{ \{ z \} \times T_z : z \in N_G(x) \} \subseteq N_{G[H]}((y,c)) \cap C \). Since \( C \) is a differentiating set in \( G[H] \), \( N_{G[H]}[x,a] \cap C \neq N_{G[H]}[y,c] \cap C \). This implies that there exists \((y,d) \in \{ y \} \times T_y \) such that \((y,d)(y,c) \in E(G[H]) \). This means that \( d \in T_y \) and \( cd \in E(H) \). Thus, \( T_y \) is a dominating set in \( H \).

For the converse, suppose that the conditions \((i),(ii),(iii)\) and \((iv)\) hold. Let \((x,a) \in V(G[H]) \). Since \( G \) is non-trivial and connected, there exists \( y \in V(G) \) such that \( xy \in E(G) \). Now, since \( S = V(G) \), there exists \( b \in V(H) \) such that \((y,b) \in C \). Thus, \((x,a)(y,b) \in E(G[H]) \). Therefore, \( C \) is a total dominating set in \( G[H] \). Now let \((x,a),(y,b) \in V(G[H]) \) with \((x,a) \neq (y,b) \). We consider the following cases. Case 1: Suppose \( x = y \). Then \( a \neq b \). By \((ii)\), \( T_x \) is a differentiating set in \( H \), hence, \( N_H[a] \cap T_x = A \neq B = N_H[b] \cap T_y \). Suppose that \( c \in A \setminus B \). Then \((x,c) \in N_{G[H]}[x,a] \setminus N_{G[H]}[y,b] \). Consequently \( N_{G[H]}[(x,a)] \cap C \neq N_{G[H]}[(y,b)] \cap C \). Therefore \( C \) is a differentiating set in \( G[H] \).

Case 2: Suppose \( x \neq y \). Consider the following subclaims.

Subclaim 1: Suppose \( xy \notin E(G) \).

If \( N_G[x] \neq N_G[y] \), then \( N_{G[H]}[x,a] \cap C \neq N_{G[H]}[y,b] \cap C \). Suppose \( N_G(x) = N_G(y) \). By \((iv)\), \( T_x \) or \( T_y \) is a dominating set in \( H^* \). Assume without loss of generality that \( T_x \) is a dominating set in \( H^* \). Then for every \( a \in V(H) \setminus T_x \) there exists \( c \in T_x \) such that \( ac \in E(H) \). It follows that \((x,c) \in C \) and \((x,a)(x,c) \in E(G[H]) \). Since \( xy \notin E(G) \), \((x,c)(y,b) \notin E(G[H]) \). It follows that

\[
N_{G[H]}[(x,a)] \cap C \neq N_{G[H]}[(y,b)] \cap C.
\]

Subclaim 2: Suppose \( xy \in E(G) \).

If \( N_G[x] \neq N_G[y] \), then \( N_{G[H]}[x,a] \cap C \neq N_{G[H]}[y,b] \cap C \). Suppose \( N_G(x) = N_G(y) \). By \((iii)\), \( T_x \) or \( T_y \) is strictly differentiating in \( H \). Assume without loss of generality that \( T_x \) is a strictly differentiating set in \( H \). Thus, for every \( d \in T_x \), \( N_H[d] \cap T_x \neq T_x \). Then, there exists \((x,d) \in C \) such that \((x,d)(x,a) \notin E(G[H]) \). Since \( xy \in E(G) \), \((x,d)(y,b) \in E(G[H]) \). It follows that

\[
N_{G[H]}[(x,a)] \cap C \neq N_{G[H]}[(y,b)] \cap C.
\]

Accordingly, \( C \) is a differentiating total dominating set in \( G[H] \).

The following is a direct consequence of Theorem 4.1.

**Corollary 4.2** Let \( G \) be a totally point determining graph of order \( n \geq 3 \) and \( H \) be a point distinguishing graph of order \( m \geq 3 \) with \( \Delta(H) \leq m - 2 \). Then \( C = \bigcup_{x \in S} \{ x \} \times T_x \) is a minimum differentiating total dominating set in \( G[H] \).
if and only if \( S = V(G) \) and \( T_x \) is a minimum differentiating set in \( H \) for every \( x \in V(G) \).

**Corollary 4.3** Let \( G \) be a totally point determining graph of order \( p \geq 3 \) and \( H \) be a point distinguishing graph of order \( m \geq 3 \) with \( \Delta(H) \leq m - 2 \). Then \( \gamma_{DT}(G[H]) = pdn(H) \).

**Proof:** Let \( C = \bigcup_{x \in S} (\{x\} \times T_x) \) be a minimum differentiating total dominating set in \( G[H] \). Then by Corollary 4.2, \( S = V(G) \) and \( T_x \) is a minimum differentiating set in \( H \) for every \( x \in V(G) \). Therefore \( \gamma_{DT}(G[H]) = |C| = pdn(H) \). \( \square \)

**References**


Received: May 1, 2014