On Some Properties Related to
Order Dual of an $f$-Module

Serap Özcan and Ömer Gök
Yıldız Technical University, Faculty of Arts and Science
Department of Mathematics, Esenler, Davutpaşa, Istanbul, Turkey

Abstract
In this paper, we investigated some studies of Y. Feng etc. [3] which are on $f$-algebras, in $f$-modules.

Mathematics Subject Classification: 46H25, 06F25

Keywords: $f$-algebra, $f$-linear operator, $f$-module, $f$-orthomorphism, order dual

1 Introduction

First of all, for unexplained terminology and notation, we refer to the standart books [1,9,11]. Let $L$ and $M$ are two Archimedean Riesz spaces and $A$ be an $f$-algebra, that is, a lattice ordered algebra in which $a \wedge b = 0$ implies that $a, c \wedge b = 0$ for all $0 \leq c \in A$ [6]. A linear operator between two Riesz spaces is said to be order bounded if it maps order bounded subsets of $L$ to order bounded subsets of $M$. The collection of all order bounded operators will be denoted by $L_b(L, M)$. In a Riesz space $L$, $x$ and $y$ are said to be disjoint if $|x| \wedge |y| = 0$ holds for $x, y \in L$. An operator $T: L \rightarrow L$ is said to be band preserving whenever $T$ leaves all bands of $L$ invariant, i.e., whenever $T(B) \subseteq B$ holds for each band $B$ of $L$. Orth($L$) is a space of all order bounded and band preserving operators on $L$ and the set of orthomorphisms on $L$ is Orth($L$) = \{T \in L_b(L); x \perp y \Rightarrow Tx \perp y\}.

$T \in L_b(L, M)$ is called an $f$-linear operator if $T(a.x) = a.Tx$ for each $a \in A$ and $x \in L$. The collection of all $f$-linear operators will be denoted by $L_b(L, M; A)$ [10].
The vector space $L'$ of all order bounded linear functionals on $L$ is called the order dual of $L$, i.e., $L' = L_b(L,R)$. $L'$ is a Riesz space [1]. The order bidual $L''$ of $L$ is the order dual of $L'$ that is $L'' = (L')'$ and the order bounded part of the order bidual of $L$ is denoted by $(L')^*_n$. A Riesz algebra $A$ is called an $f$-algebra if $A$ has the additional property that $a \perp b$ in $A$ implies $ca \perp b$ and $ac \perp b$ for all $c \in A$. Since every Archimedean $f$-algebra is commutative [7], we deal only with commutative $f$-algebras in this work. Let $A$ be an $f$-algebra with $A'$ separates the points of $A$. The band of all order continuous linear functionals on $A'$ is denoted by $(A')^*_n$ and its disjoint complement in $A''$ by $(A')^*_s$. Observe that, $A'' = (A')^*_n \oplus (A')^*_s$ as $A''$ is Dedekind complete [4,5].

**Definition 1** Let $A$ be an $f$-algebra with unit $e$ and $L$ be a Riesz space. $L$ is said to be an (left) $f$-module over $A$ [8,10] if $L$ is a (left) module over $A$ and satisfies the following conditions:

i) For each $a \in A^+$ and $x \in L^+$ we have $a.x \in L^+$

ii) If $x \perp y$ in $L$, then for each $a \in A$ we have $a.x \perp y$

When $A$ is an $f$-algebra with unit $e$, saying $L$ is a unital $f$-module over $A$ implies that the left multiplication satisfies $e.x = x$ for all $x \in L$. We know that if $L$ is an $f$-module over $A$, then $L'$ is an $f$-module over $A$ (and $A''$) [3,10].

Throughout the paper, we only consider Archimedean Riesz spaces with point separating order dual.

**Lemma 2** Let $A$ be an unital $f$-algebra and $L$ be a Riesz space. If $L$ is an $f$-module over $A$, we can define the mappings [2,5]:

1) $L \times L' \to A'$

$(x,f) \to (x.f) : (x.f)(a) = f(a.x)$ for $x \in L, f \in L', a \in A$,

2) $A'' \times L' \to L''$

$(F,f) \to T.f : (F.f)(x) = F(f.x)$ for $x \in L, f \in L', F \in A''$,

3) $A'' \times L'' \to L''$

$(F,f) \to (F.f)' : (F.f)'(f) = f(F.f)$ for $f \in L', F \in A''$, $\hat{f} \in L''$.

A multiplication can be introduced on $A''$, by mapping 3, the order bidual of $A$, with respect to which $A''$ can also be made an $f$-algebra. Let $A$ be an unital $f$-algebra with separating order dual, $L$ be an $f$-module over $A$, $f \in L'$ and $F \in A''$. We consider the mapping $T_f : A'' \to L'$ defined by $T_f(F) = F.f$ for all $F \in A''$. Note that the mapping $V : A'' \to \text{Orth}(L')$ defined by $V(F) = V_F$ for all $F \in A''$, where $V_F(f) = F.f$ for every $f \in L'$, is linear, positive, algebra and Riesz homomorphism [10,5]. Hence, we get the following theorem.

**Theorem 3** [3] Let $L$ be an $f$-module over an $f$-algebra $A$. For $0 \leq f \in L'$, the mapping $T_f : A'' \to L'$ defined by $T_f(F) = F.f$ for all $F \in A''$ is a lattice homomorphism.

**Proof**: $T_f$ is positive. So, we will prove the following claims respectively.
On some properties related to order dual of an $f$-module

i) $T_f$ is a linear mapping.

ii) $T_f$ is lattice homomorphism.

i) $T_f : A'' \rightarrow L'$

$T_f(F) = F \cdot f$ for $F \in A''$, $f \in L'$.

$T_f$ is a linear mapping:

a) $T_f(F + G) = T_f(F) + T_f(G)$. For this, we must show that

$T_f(F + G)x = (T_f(F))x + (T_f(G))x$ is true for all $x \in L$.

$T_f(F + G)x = ((F + G), f)x$

$= (F + G)(f, x)$

$= F(f, x) + G(f, x)$

$= (F \cdot f)x + (G \cdot f)x$

$= (T_f(F))x + (T_f(G))x$.

b) $T_f(\alpha F) = \alpha (T_f(F))$. To prove this, we must show that

$(T_f(\alpha F))x = \alpha (T_f(F))x$ is true for all $x \in L$.

$(T_f(\alpha F))x = ((\alpha F), f)x$

$= (\alpha F)(f, x)$

$= (\alpha F \cdot f)(x)$

$= \alpha (F \cdot f)x$

$= \alpha (T_f(F))x$.

So, $T_f$ is a linear mapping.

ii) Since the mapping $V : A'' \rightarrow Orth(L')$ is a lattice homomorphism and $V_F, V_G \in Orth(L')$ for $F, G \in A''$, we get the following:

$T_f(F \lor G) = (F \lor G), f$

$= V_{F \lor G}(f)$

$= (V(F \lor G))(f)$

$= (V(F) \lor V(G))(f)$

$= (V(F)(f)) \lor (V(G)(f))$

$= F \cdot f \lor G \cdot f$

$= T_f(F) \lor T_f(G)$

So, $T_f$ is a lattice homomorphism.

**Theorem 4** [3] Let $L$ be an $f$-module over an $f$-algebra $A$. For all $f \in L'$ and $F \in A''$, we get $|F \cdot f| = |F| \cdot |f|$. Also, if $f \perp g$ in $L'$, then $F \cdot f \perp G \cdot g$ holds for each $F, G \in A''$.

**Proof:** Let us take $f \in L'$. Decompose $f = f^+ - f^-$. Since $f^+ \land f^- = 0$ and $V_F \in Orth(L')$, we have $V_F(f^+) \perp V_F(f^-)$. From Theorem 3 and by using the formulas, $|f| = f^+ + f^-$ and $f = f^+ - f^-$, we obtain,

$|F \cdot f| = |F \cdot f^+| + |F \cdot f^-|

= |T_{f^+}(F)| + |T_{f^-}(F)|$
Let \( f \perp g \) in \( L' \). Then we have,
\[
|F \cdot f| \wedge |G \cdot g| = |F| \cdot |f| \wedge |G| \cdot |g|
\leq ((|F| + |G|) \cdot |f|) \wedge ((|F| + |G|) \cdot |g|)
= (|F| + |G|) \cdot (|f| \wedge |g|)
= 0
\]
which implies that \( F \cdot f \perp G \cdot g \) for all \( F, G \in A'' \).

Denote the image of \( A'' \) under \( T_f \) by \( R(f) = \{ F : F \in A'' \} \). \( R(f) \) is a linear subspace of \( L' \).

**Definition 5** Let \( A \) be an Archimedean \( f \)-algebra. Suppose that \( L \) is an \( f \)-module over \( A \) and \( f \in L' \). \( L \) is said to have the \( k \)-property if and only if \( f \cdot a = 0 \) for each \( a \in A \), then \( f = 0 \).

**Theorem 6** [3,10] Let \( A \) be an \( f \)-algebra and \( L \) be an \( f \)-module with the \( k \)-property over \( A \). If \( f, g \in L' \), then \( f \perp g \) if and only if \( R(f) \perp R(g) \).

**Proof:** Let \( f \perp g \) in \( L' \). We know from Theorem 4 that, \( F \cdot f \perp G \cdot g \) for all \( F, G \in A'' \). This implies that \( R(f) \perp R(g) \).

Conversely, if \( R(f) \) and \( R(g) \) are disjoint, then for each \( F \in (A'')^+ \), we have
\[
F \cdot (|f| \wedge |g|) = V_F(|f| \wedge |g|)
= V_F(|f|) \wedge V_F(|g|)
= |F| \cdot |f| \wedge |F| \cdot |g|
= |F| \cdot |f| \wedge |F| \cdot |g|
= 0.
\]

In particular, for any \( a \in A \), its canonical image \( a'' \in A'' \) also satisfies
\[
a''.(|f| \wedge |g|) = (|f| \wedge |g|) \cdot a = 0.
\]
By the preceding definition, we have \( |f| \wedge |g| = 0 \), which implies that \( f \perp g \).

Let \( A \) be a unital \( f \)-algebra with separating order dual, \( L \) be a \( f \)-module over \( A \) and \( T \in L_b(L') \). Recall that \( T \) is said to be \( f \)-linear with respect to \( A'' \) if \( T(G \cdot f) = G \cdot (T(f)) \) for all \( f \in L' \) and \( G \in A'' \). We will denote by \( L_b(L', L'' ; A'') \), the set of all \( f \)-linear operators on \( L' \).

**Theorem 7** [3,10] Let \( A \) be a unital \( f \)-algebra with separating order dual, \( L \) be an \( f \)-module over \( A \). Then, we get \( Orth(L') \subseteq L_b(L', L'' ; A'') \).

**Proof:** It is clear that \( Orth(L') \) is commutative. Let us take \( \pi \in Orth(L') \). We have \( \pi(G \cdot f) = \pi(V_G(f)) = V_G(\pi(f)) = G \cdot (\pi(f)) \), for all \( f \in L' \) and \( G \in A'' \). Therefore, \( \pi \in L_b(L', L'' ; A'') \).
On some properties related to order dual of an f-module

References


Received: May 1, 2014