About the Solution in Closed Form of Generalized Riemann Boundary Value Problem with Shift in the Class of Analytical Functions

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Abstract

The generalized Riemann boundary value problem with shift is investigated in the class of piecewise analytic functions. In the case where the coefficients of the problem satisfy some equalities, the algorithm for the solution of this problem was obtained, and its solution in closed form are determined.

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1 Introduction

Let $L$ is a simple closed Lyapunov curve dividing the closed complex plane into the interior part $D^+$ and exterior part $D^-$, $0 \in D^+$, $\alpha(t)$ is a homeomorphism onto itself which preserves or changes the orientation of $L$. 
Find function \( \Phi^+(z) \) and \( \Phi^-(z) \) analytic in \( D^+ \) and \( D^- \), respectively, satisfying the condition
\[
\Phi^+[\alpha(t)] = G_1(t)\Phi^-(t) + G_2(t)\overline{\Phi^-(t)} + f(t), \quad t \in L
\]
(1)
imposed on their boundary values on the contour \( L \).

The problem above is called generalized Riemann boundary value problem with shift\([4]\) or is called generalized Markushevich boundary value problem\([9]\). And it should be noted that the problem in form (1) in case \( \alpha(t) \equiv t \) firstly was formulated in 1946 by A.I.Markushevich\([5]\).

During the last 60 years many original works\([2,3,6,8,10,12]\) have been devoted to the problem (1). Even in the first works\([1,11]\), devoted to the investigation of the problem (1) it was established that if the condition
\[
G_1(t) \neq 0, \quad t \in L
\]
is fulfilled, it is the Noether problem. And G.S.Litvinchuk\([4]\) reviewed the survey of closely related results of problem (1), then the Noether theory, stability, and solvability theory were all mentioned.

In this article we shall obtain the constructive algorithm for solution of the problem (1), and when one case of \( G_1(t) \pm \overline{G_2(t)} \equiv const \) is satisfied, the problem can be solved in a closed form.

2 The solution of problem (1) in a close form

Let the curve \( L \) be the unit circle, i.e. \( D^+ = \{ z : |z| < 1 \} \), \( D^- = \{ z : |z| > 1 \} \), and \( \alpha(t) \) is a direct shift, \( G_1(t), G_2(t), f(t) \) are given on \( L \) functions of Holder class and \( G_1(t) \neq 0 \) on \( L \). When one case of \( G_1(t) \pm \overline{G_2(t)} \equiv const \) is satisfied, we shall solve the problem (1) under the condition \( \Phi^-(\infty) = 0 \).

Denote \( \kappa = ind[\arg G_1(t)] \), and without loss of generality it may be supposed that \( |G_1(t)| = 1 \).
Assuming that $G_i(t) + G_j(t) = a + bi, a, b \in R$, then, by taking conjugates on both sides of problem (1),

$$\Phi^+[\alpha(t)] = G_i(t)\Phi^-(t) + G_j(t)\Phi^+(t) + f(t), \quad t \in L$$

(2)

Then by (1) and (2),

$$\Phi^+[\alpha(t)] + \Phi^-[\alpha(t)] = (a - bi)\Phi^-(t) + (a + bi)\Phi^+(t) + 2\Re[f(t)], \quad t \in L$$

(3)

Let

$$\begin{cases}
\Psi^+(z) = \Phi^+[\alpha(z)] - (a - bi)\Phi^-(\frac{z}{\alpha}), & z \in D^+ \\
\Psi^-(z) = (a + bi)\Phi^-(z) - \Phi^+[\alpha(\frac{1}{z})], & z \in D^- 
\end{cases}$$

(4)

Because of $\alpha(t)$ is a direct shift, and $L$ is a unit circle, then function $\Psi^+(z), \Psi^-(z)$ can analytic in $D^+$ and $D^-$, respectively, and $\psi(\infty) < \infty$. So (3) may be rewritten as

$$\Psi^+(t) - \Psi^-(t) = 2\Re[f(t)]$$

(5)

Then the general solution of (5) in $R_0$ is

$$\Psi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{2\Re[f(t)]}{t - z} dt + C$$

(6)

where $C$ is an arbitrary constant. By Plemelj formula, we have

$$\begin{cases}
\Psi^+(t) = \Phi^+[\alpha(t)] - (a - bi)\Phi^-(t) = \Re[f(t)] + \frac{1}{2\pi i} \int_{\Gamma} \frac{2\Re[f(\tau)]}{\tau - t} d\tau + C \\
\Psi^-(t) = (a + bi)\Phi^-(t) - \Phi^+[\alpha(t)] = -\Re[f(t)] + \frac{1}{2\pi i} \int_{\Gamma} \frac{2\Re[f(\tau)]}{\tau - t} d\tau + C
\end{cases}$$

(7)

Substituting (7) into problem (1), we can get

$$\Im[G_i(t)\Phi^-(t)] = -\frac{iC}{2} - \frac{1}{4\pi} \int_{\Gamma} \frac{2\Re[f(\tau)]}{\tau - t} d\tau - \frac{\Im f(t)}{2}$$

(8)

Because of $|G_i(t)| = 1$ is satisfied, and we define

$$u(t) := -\frac{iC}{2} - \frac{1}{4\pi} \int_{\Gamma} \frac{2\Re[f(\tau)]}{\tau - t} d\tau - \frac{\Im f(t)}{2}$$

, then (8) maybe rewritten in the Riemann-Hilbert outer problem.
Suppose that $\beta(t)$ is an inverse homeomorphism of $\alpha(t)$. By formula (4), we have

$$\Phi^+(z) = \Psi^+[\beta(z)] + (a - bi)\Phi^-\left[\frac{1}{\beta(z)}\right]$$  \hspace{1cm} (10)

Obviously, solving the problem (1) is equivalent to solving the problem (10) and the Riemann-Hilbert outer problem (9). Therefore, we have

1) If $\kappa \geq 0$, the solution of problem (9) as follows

$$\Phi^-(z) = z^{-\kappa}e^{(r(z)[S(|t|^{\kappa}e^{\alpha(t)} u(s)) + Q(z))]$$  \hspace{1cm} (11)

where $\gamma(z) = \text{Re} \frac{G_i(t)}{G_j(t)} + \kappa \text{arg} t$, $\omega_i(s) = \text{Im} \gamma(z)$. $S$ denotes a Schwarz operator, which analytic in $D^-$, and its imaginary part is equal to zero on infinity. $Q(z)$ analytic in $D^-$ except infinity, and it can be written as follows

$$Q(z) = i\beta_0 + \sum_{k=1}^{\kappa} \left(c_k z^{-k} - \overline{c_k} z^k\right)$$  \hspace{1cm} (12)

Substituting (6) and (11) into (10), we have

$$\Phi^+(z) = \frac{1}{2\pi i} \int_{t}^{2\pi} \text{Re}[f(t)] dt + C + (a - bi)\Phi^-\left[\frac{1}{\beta(z)}\right]$$  \hspace{1cm} (13)

2) If $\kappa < 0$, and the following conditions are satisfied

$$\int_{0}^{2\pi} e^{i\omega_i(s)} u(s)e^{-\kappa \sigma}d\sigma = 0 \hspace{1cm} (k = 0, 1, \ldots, -\kappa - 1)$$  \hspace{1cm} (14)

The (9) is solvable and has a unique solution

$$\Phi^-(z) = z^{-\kappa}e^{(r(z)[S(|t|^{\kappa}e^{\alpha(s)} u(s))]$$  \hspace{1cm} (15)

Substituting (6) and (15) into (13), we obtain

$$\Phi^+(z) = \frac{1}{2\pi i} \int_{t}^{2\pi} \text{Re}[f(t)] dt + C + (a - bi)\Phi^-\left[\frac{1}{\beta(z)}\right]$$  \hspace{1cm} (16)

Thus, we get
Theorem 1. Suppose that the problem (1) satisfied the condition \( G_1(t) + \overline{G_2(t)} = \text{const} \). 1) If \( \kappa \geq 0 \), then the boundary value problem (1) is solvable unconditionally, and its general solution is represented by formula (11) and (13), and depends on \(-2\kappa + 1\) arbitrary real constants. 2) If \( \kappa < 0 \), then problem (1) is solvable and has a unique solution if and only if \( 2\kappa - 1 \) real solvability conditions (14) hold, and the solutions are expressed in an explicit form by formulas (15) and (16).

Similarly, we have

Theorem 2. Suppose that the problem (1) satisfied the condition \( G_1(t) - \overline{G_2(t)} = \text{const} \), we can get a similar conclusion.

Example 1. Let \( D^+ = \{ z : |z| < 1 \} \), \( D^- = \{ z : |z| > 1 \} \) and \( L = \{ t : |t| = 1 \} \). It is required to find functions \( \Phi^+(z) \) and \( \Phi^-(z) \) analytic in \( D^+ \) and \( D^- \), respectively, which vanishing on the infinity and satisfying on \( L \) the following boundary condition

\[
\Phi^+(t) = t^2\Phi^-(t) - \frac{1}{t^2} + 3\Phi^-(t) + 4, \quad |t| = 1
\]  

(17)

Solution. Here \( \alpha(t) = -t \), \( G_1(t) = t^2 \), \( G_2(t) = -\frac{1}{t^2} - 3 \), \( f(t) = 4 \), then \( G_1(t) + \overline{G_2(t)} = -3 \). Hence, taking into consideration formulas (6) and (7), we get in this case

\[
\kappa = \text{Ind}(t^2) = 2, \quad u(t) = \frac{C_0}{2}
\]

\[
\Psi^+(z) = 4 + iC_0
\]

\[
\Psi^-(z) = -4 + iC_0
\]

Where \( C_0 \) is an arbitrary real constant. Then the following functions will be the solution of the problem (17)
\[
\Phi^-(z) = \frac{C_0}{2z^2} i \\
\Phi^+(z) = 4 + iC_0 + \frac{3i}{2} C_0 z^2
\]

3 Some special cases

3.1 \( G_1(t) + \overline{G_2(t)} = 0 \)

Then the problem (1) can change into a degenerate problem, and it can be written as follows

\[
\begin{align*}
\text{Re} \Phi^+[\alpha(t)] &= \text{Re} f(t) \\
\text{Im} G_1(t) \Phi^- (t) &= \frac{i}{2} f(t) - \frac{i}{2} \Phi^+[\alpha(t)]
\end{align*}
\]

Then the closed form solution of problem (1) is obtained.

Remark: here \( \alpha(t) \) can be a direct or an inverse shift.

3.2 \( G_1(t) - \overline{G_2(t)} = 0 \)

Similarly, the problem (1) can change into a degenerate problem, and it can be written as follows

\[
\begin{align*}
\text{Im} \Phi^+[\alpha(t)] &= \text{Im} f(t) \\
\text{Re} G_1(t) \Phi^- (t) &= \frac{\Phi^+[\alpha(t)] - f(t)}{2}
\end{align*}
\]

Then the closed form solution of problem (1) is obtained.

Remark: here \( \alpha(t) \) can be a direct or an inverse shift.

3.3 In 1992 V. Mityushev\(^7\) discussed a boundary value problem \( \Phi^+(t) = \Phi^-(t) + \lambda \Phi^-(t) \), and an equivalent operator equation was obtained. Obviously, the above problem in is a special case of this paper, so it can be obtained as an immediate consequence of our results.

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