Homotopy Analysis Method for Cauchy Riemann Equations

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Abstract

In this paper, we obtain the series solution for the Cauchy Riemann equations with initial value problem by using homotopy analysis method (HAM), and computer graphics show that the HAM is efficient in solving Cauchy - Riemann equations.

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1 Introduction

The HAM was first proposed by Liao in 1992 [1]. This method has been successfully applied to solve various linear or nonlinear problems [6], [2], [3], [5], [4]. In this paper, we use this method to solve Cauchy Riemann equations, which are given by

\[
\begin{align*}
    v_t(x, t) + \rho_x(x, t) &= 0, \\
    \rho_t(x, t) - v_x(x, t) &= 0,
\end{align*}
\]

(1)

2 HAM

In this section, we consider a linear or nonlinear equation in a general form:

\[ N[u(x, t)] = 0, \] (2)
where \( u(x, t) \) is an unknown function, \( x \) and \( t \) are independent variables. Let \( u_0(x, t) \) denote an initial approximation of the solution of equation (11), \( h \) a nonzero auxiliary parameter, \( H(x, t) \) a nonzero auxiliary function and \( L \) an auxiliary linear operator. Then we construct the HAM deformation equation in the following form:

\[
(1 - q)L[\Phi(x, t; q) - u_0(x, t)] = qhH(x, t)N[\Phi(x, t; q)],
\]

where \( q \in [0, 1] \) is an embedding parameter. Obviously, when \( q = 0 \) and \( q = 1 \), the above HAM deformation equation (12) has the solutions

\[
\Phi(x, t; 0) = u_0(x, t), \quad \Phi(x, t; 1) = u(x, t),
\]

respectively. Thus as \( q \) increases from 0 to 1, \( \Phi(x, t; q) \) varies from the initial guesses \( \Phi(x, t; 0) \) to the solution \( \Phi(x, t; 1) \) of equation (11). Expanding \( \Phi(x, t; q) \) in Taylor's series with respect to \( q \), we have

\[
\Phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m,
\]

where

\[
u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \Phi(x, t; q)}{\partial q^m} \right|_{q=0}.
\]

For brevity, define a vector

\[
\vec{u}_m = \{u_0, u_1, \ldots, u_m\}.
\]

Differentiating the HAM deformation equation (12) \( m \) times with respect to \( q \), then setting \( q = 0 \), and dividing it by \( m! \), we obtain the \( m \)th-order deformation equation

\[
L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = hH(x, t)R_m(\vec{u}_{m-1}(x, t)),
\]

where \( R_m(\vec{u}_{m-1}(x, t)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\Phi(x, t; q)]}{\partial q^{m-1}} \right|_{q=0} \), and

\[
\chi_m = \begin{cases} 0, & m = 1, \\ 1, & m > 1. \end{cases}
\]

Operating the inverse operator of \( L \) on the both sides of equation (13), we have

\[
u_m(x, t) = \chi_m u_{m-1}(x, t) + hH(x, t)L^{-1}R_m(\vec{u}_{m-1}(x, t)).
\]

In this way, it is easy to obtain \( u_1(x, t), u_2(x, t), \ldots \), one after another, finally, we get an exact solution of the original equation (11)

\[
u(x, t) = \sum_{m=0}^{\infty} u_m(x, t).
\]
3 Applying HAM

In this section, we consider fractional Cauchy Riemann equations (1) with the following initial conditions

\[ v(x, 0) = \sin(x) = v_0(x, t), \quad \rho(x, 0) = \cos(x) = \rho_0(x, t). \] (5)

First, we choose two linear fractional order operators

\[ L_1[\varphi(x, t; q)] = \frac{\partial \varphi(x, t; q)}{\partial t}, \quad L_2[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t}. \]

Secondly, we define two linear operators as

\[ N_1[\varphi(x, t; q), \phi(x, t; q)] = \frac{\partial \varphi(x, t; q)}{\partial t} + \frac{\partial \phi(x, t; q)}{\partial x}, \]

\[ N_2[\phi(x, t; q), \varphi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t} - \frac{\partial \varphi(x, t; q)}{\partial x}. \]

Using above definitions, and with assumption \( H(x, t) = 1 \), we obtain

\[ \begin{align*}
v_0(x, t) &= \sin(x), \\
\rho_0(x, t) &= \cos(x), \\
\end{align*} \]

\[ \begin{align*}
v_1(x, t) &= -hsin(x), \\
\rho_1(x, t) &= -hcos(x), \\
\end{align*} \]

\[ \begin{align*}
v_2(x, t) &= -hsin(x)(1 + h - \frac{1}{2}hx), \\
\rho_2(x, t) &= -hcos(x)(1 + h - \frac{1}{2}hx), \\
\end{align*} \]

\[ \begin{align*}
v_3(x, t) &= -hsin(x)(1 + 2h - hx + h^2 - h^2x^2 - \frac{1}{6}hx^2), \\
\rho_3(x, t) &= -hcos(x)(1 + 2h - hx + h^2 - h^2x^2 - \frac{1}{6}hx^2), \\
\end{align*} \]

\[ \ldots \]

We set \( h = -1 \), then we have

\[ v(x, t) = e^t\sin(x), \quad \rho(x, t) = e^t\cos(x). \] (6)
Figure 1: The exact solution of $v(x, t)$ and $\rho(x, t)$.

Figure 2: The approximate solution of $v(x, t)$ and $\rho(x, t)$.

References


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