Another Method to Evaluate Green Functions for Elliptic Equations

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Abstract

It is well known that the resolvent kernel for a strongly elliptic operator with constant coefficients, which is the inverse Fourier transform of the related symbol, satisfies the exponential decay estimate. We derive it by the formula that expresses the resolvent in terms of the integration with respect to spectral parameter of a high power of the resolvent, whose kernel can be evaluated by the standard method. This approach also works when the elliptic operator is defined on a domain and has variable coefficients.

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1 Green functions for constant coefficients

Let $m$ be an even positive integer and let a homogeneous polynomial of $\xi \in \mathbb{R}^n$

$$a(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

satisfy the strong ellipticity condition

$$a(\xi) \geq \delta |\xi|^m \quad \text{for all } \xi \in \mathbb{R}^n$$ \hspace{1cm} (1)
with some $\delta > 0$. For a positive integer $k$, a multi-index $\alpha$ and a complex number $\lambda \in \mathbb{C} \setminus [0, \infty)$ we set

$$G^\alpha_k(x, \lambda) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} \frac{\xi^\alpha}{(a(\xi) - \lambda)^k} d\xi$$

(2)

with $i = \sqrt{-1}$, where the integral should be interpreted as the inverse Fourier transform of a tempered distribution if $mk \leq n + |\alpha|$. More precisely, if we denote by $\langle T, f \rangle$ the value of a tempered distribution $T$ at $f \in \mathcal{S}(\mathbb{R}^n)$, then the tempered distribution $G^\alpha_k(x, \lambda)$ is defined by

$$\langle G^\alpha_k(\cdot, \lambda), f \rangle = \int_{\mathbb{R}^n} \frac{\xi^\alpha}{(a(\xi) - \lambda)^k} \mathcal{F}^{-1} f(\xi) \ d\xi$$

(3)

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Here the Fourier transformation $\mathcal{F}$ is defined by $\mathcal{F} f(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x) \ dx$, and the inverse of $\mathcal{F}$ is given by $\mathcal{F}^{-1} f(x) = c_n \int_{\mathbb{R}^n} e^{ix\xi} f(\xi) \ d\xi$ with $c_n = (2\pi)^{-n}$.

If $k = 1$ and $\alpha = 0$, it is known that the function $G^0_1(x, \lambda)$ is the fundamental solution of the operator $a(D) - \lambda$ or the Green function of the elliptic equation $(a(D) - \lambda)u = f$ for a given function $f$. Here $D = (D_1, \ldots, D_n)$ with $D_j = i^{-1} \partial / \partial x_j$ ($j = 1, \ldots, n$).

It is also known that $G^0_1(x, \lambda)$ is an $L_1$-function satisfying the exponential decay estimate, which will be formulated as (5) below. To the best of our knowledge, there are several ways to derive estimate (5). Gurarie [5] used the result for pseudodifferential operators of negative order (see [4]) after changing the contour of integration. The key step used in Shimakura [10], Denk, Hieber and Prüss [2, Section 5.1] and Miyazaki [9] is the Fubini type identity

$$G^0_1(x, \lambda) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} e^{i|\xi_1|} \right) \ d\xi_1$$

(4)

with $\xi = (\xi_1, \xi')$ for $x \neq 0$, where $U_x$ is a rotation in $\mathbb{R}^n$ satisfying $U_x(e_1) = x/|x|$ with $e_1 = (1, 0, \ldots, 0)$. In [10], which deals with a wider class of elliptic operators, the justification of (4) is omitted. In [2] identity (4) is derived by multiplying the integrand in (2) by the factor $e^{-i|\xi|}$ with $\epsilon > 0$ and deforming the integral path to the contour defined by $z = t + i\kappa(1 + |t|)$, $t \in \mathbb{R}$ with some constant $\kappa > 0$. In [9] the function $\exp(-\epsilon|\xi|^2)$ is used as the convergence factor in (2), and (4) is derived by evaluating the convolution of the Fourier transform of $\exp(-\epsilon|\xi'|^2)$ and the $\xi_1$-integral in (4). It would also be possible to combine the Gaussian estimate for $\exp(-t\alpha(D))$ obtained in [3, Chapter 9, Theorem 1] and relation (12) below.

In this paper, we propose another approach to obtain the exponential decay estimate of $G^0_k(x, \lambda)$ when $m \leq n$. For our purpose it is convenient to consider $G^\alpha_k(x, \lambda)$ for the general $k$ and $\alpha$. We show that the exponential decay estimate
for $mk \leq n + |\alpha|$, in which case the integrand in (2) is not an $L^1$-function, can be derived from that for $mk > n + |\alpha|$, which was essentially obtained by Hörmander [6, Lemma 4.6].

In order to formulate the exponential decay estimate, we define the function $\log_+ t$ by $\log_+ t = \max \{\log t, 0\}$ for $t > 0$, and set

$$
\Phi(x, \sigma, c) = \exp(-c|x|) \times \begin{cases} 1 & (\sigma > 0) \\
1 + \log_+ |x|^{-1} & (\sigma = 0) \\
|x|^\sigma & (\sigma < 0)
\end{cases}
$$

for $x \in \mathbb{R}^n$, $\sigma \in \mathbb{R}$ and $c > 0$. For $\omega > 0$ we set

$$
\Lambda_\omega = \{\lambda \in \mathbb{C} : \omega \leq \text{arg} \lambda \leq 2\pi - \omega, \lambda \neq 0\}.
$$

Let $M = \max \{|a_\alpha| : |\alpha| = m\}$.

**Theorem 1.1.** Let $\omega \in (0, \pi/2)$ and $\lambda \in \Lambda_\omega$. Let a positive integer $k$ satisfy $mk > n + |\alpha|$. Then the tempered distribution $G_k^\alpha(x, \lambda)$ belongs to $L_1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{0\})$, and there exist positive constants $C$ and $c$ depending only on $m$, $n$, $\delta$, $M$ and $\omega$ such that the inequality

$$
|G_k^\alpha(x, \lambda)| \leq C|\lambda|^{-k+(n+|\alpha|)/m} \Phi(|\lambda|^{1/m}x, mk - n - |\alpha|, c) \tag{5}
$$

holds. Moreover, $c$ can be written as $c = c_0 \sin \omega$ with some constant $c_0$ depending only on $m$, $n$, $\delta$ and $M$.

For the proof of Theorem 1.1 we need two lemmas. The first one is the key tool of our method, and the second one is merely a precise formulation of the inequality used in [5, 6].

**Lemma 1.2.** Let $k$ and $l$ be positive integers, and set $c_{k,l} = (-1)^l k^{(k+l-1)}$. Then

$$
(s - \lambda)^{-k} = c_{k,l} \int_{L(\lambda)} (\mu - \lambda)^{l-1}(s - \mu)^{-k-l} d\mu
$$

holds for $s > 0$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$, where $L(\lambda)$ stands for the half line parametrized by $\mu = t\lambda/|\lambda|$ with $|\lambda| \leq t < \infty$.

**Proof.** Since $\frac{d}{d\mu}(s-\mu)^{-k} = k(s-\mu)^{-k-1}$, we get the identity for $l = 1$. Repeated use of integration by parts yields the identity for $l \geq 2$. \qed

**Lemma 1.3.** Let $d(\lambda) = \text{dist} (\lambda, [0, \infty))$. There exists a positive constant $c_0$ depending only on $n$, $m$, $\delta$ and $M$ such that if $|\eta| \leq c_0 d(\lambda)$ then

$$
|a(\xi + i\eta) - \lambda| \geq \frac{d(\lambda)}{8}(a(\xi) + 1)
$$

for $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$ with $|\lambda| = 1$. 

Proof. The lemma is proved by the inequality

$$|a(\xi) - \lambda| \geq \frac{d(\lambda)}{4}(a(\xi) + 1). \quad (6)$$

See [9, Lemma 3] for the details. \qed

Proof of Theorem 1.1. First we assume $mk > n + |\alpha|$. As already stated, this case can be handled by the standard method used in [6], which originates in the proof of the Paley-Wiener theorem. For sake of completeness we give an outline of the proof for this case. We need only consider the case $|\lambda| = 1$, since

$$G_k^\alpha(x, \lambda) = |\lambda|^{-k+(n+|\alpha|)/m}G_k^\alpha(|\lambda|^{1/m}x, \lambda/|\lambda|)$$

holds by making the change of variables $\xi = |\lambda|^{1/m}\eta$ in (2). As $G_k^\alpha(x, \lambda)$ is the inverse Fourier transform of an $L_1$-function, we know that $G_k^\alpha(x, \lambda)$ is a bounded and continuous function. In view of Lemma 1.3 we can shift the whole space of integration in (2) to $\mathbb{R}^n + i\eta$ by Cauchy’s integral theorem:

$$G_k^\alpha(x, \lambda) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\xi x - \xi \eta} \frac{(\xi + i\eta)^\alpha}{(a(\xi + i\eta) - \lambda)^k} d\xi$$

for $|\eta| \leq c_0d(\lambda)$ with the constant $c_0$ in Lemma 1.3.

Setting $\eta = cx/|x|$ with $c = c_0 \sin \omega$ and using Lemma 1.3, we get $|G_k^\alpha(x, \lambda)| \leq C \exp(-c|x|)$, which implies $G_k^\alpha(\cdot, \lambda) \in L_1(\mathbb{R}^n)$. Thus for $mk > n + |\alpha|$ we obtain the assertions in Theorem 1.1 except that $G_k^\alpha(\cdot, \lambda) \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

Next, we consider the general case, not necessarily assuming $mk > n + |\alpha|$. We take a positive integer $l$ so that $(k + l)m > n + |\alpha|$. By (3) and Lemma 1.2 we have

$$\langle G_k^\alpha(\cdot, \lambda), f \rangle = c_{k,l} \int_{\mathbb{R}^n} \left( \int_{L(\lambda)} \frac{(\mu - \lambda)^{l-1}\xi^\alpha}{(a(\xi) - \mu)^{k+l}} d\mu \right) \mathcal{F}^{-1} f(\xi) d\xi$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. Since $\xi^\alpha \mathcal{F}^{-1} f(\xi) \in \mathcal{S}(\mathbb{R}^n)$ and

$$\frac{|\mu - \lambda|^{l-1}}{|a(\xi) - \mu|^{k+l}} \leq \frac{|\mu|^{l-1}}{(4^{-1}d(\mu))^{k+l}} \leq C|\mu|^{-k-1}$$

for $\mu \in L(\lambda)$ by (6), Fubini’s theorem gives

$$\langle G_k^\alpha(\cdot, \lambda), f \rangle = c_{k,l} \int_{L(\lambda)} (\mu - \lambda)^{l-1} \left( \int_{\mathbb{R}^n} \frac{\xi^\alpha \mathcal{F}^{-1} f(\xi)}{(a(\xi) - \mu)^{k+l}} d\xi \right) d\mu$$

$$= c_{k,l} \int_{L(\lambda)} (\mu - \lambda)^{l-1} \left( \int_{\mathbb{R}^n} G_{k+l}^\alpha(x, \mu) f(x) dx \right) d\mu.$$

We write \( J \) and \( | \cdot | \) for obtain the desired estimate for large, we conclude that the exponential decay estimate and hence 

\[
\int_{\mathbb{R}^n} |G_{k+l}^{\alpha}(x, \mu)| \, dx \leq C|\mu|^{-k-l+|\alpha|/m}
\]

with some \( C > 0 \). So the assumption \( mk > |\alpha| \) gives 

\[
\int_{L(\lambda)} |d\mu| \int_{\mathbb{R}^n} |(\mu - \lambda)^{l-1}G_{k+l}^{\alpha}(x, \mu)f(x)| \, dx \\
\leq C\|f\|_{L_\infty(\mathbb{R}^n)} \int_{L(\lambda)} |\mu|^{-k-l+|\alpha|/m} |d\mu| < \infty.
\]

Therefore we can apply Fubini’s theorem again to obtain \( G_k^{\alpha}(\cdot, \lambda) \in L_1(\mathbb{R}^n) \) and 

\[
G_k^{\alpha}(x, \lambda) = c_{k,l} \int_{L(\lambda)} (\mu - \lambda)^{l-1}G_{k+l}^{\alpha}(x, \mu) \, d\mu. \tag{7}
\]

Since the integral in (7) converges uniformly with respect to \( x \) outside any fixed ball centered at the origin, we know that \( G_k^{\alpha}(\cdot, \lambda) \in C(\mathbb{R}^n \setminus \{0\}) \).

By (7) we can estimate \( G_k^{\alpha}(x, \lambda) \) for \( mk \leq n + |\alpha| \). Indeed, the evaluation for \( G_{k+l}^{\alpha}(x, \lambda) \), with some \( c > 0 \), gives 

\[
|G_k^{\alpha}(x, \lambda)| \leq C \int_{L(\lambda)} |\mu - \lambda|^{l-1}|\mu|^{-k-l+(n+|\alpha|)/m} \exp(-2c|\mu|^{1/m}|x|) |d\mu| \\
\leq C \exp(-c|x|) \int_1^\infty t^{-k-l+(n+|\alpha|)/m} \exp(-ct^{1/m}|x|) \, dt.
\]

The change of variables \( t^{1/m}|x| = s \) yields 

\[
|G_k^{\alpha}(x, \lambda)| \leq C|x|^{mk-n-|\alpha|} \exp(-c|x|) \int_{|x|}^\infty s^{-m+mk+n+|\alpha|-1} e^{-cs} \, ds.
\]

We write \( J \) for the last integral. If \( mk < n + |\alpha| \), then \( J \) is bounded by a constant. If \( mk = n + |\alpha| \), then \( J \) is bounded by \( C_0 := \int_1^\infty s^{-1} e^{-cs} \, ds < \infty \) for \( |x| \geq 1 \), and by \( C_0 + \int_{|x|}^1 s^{-1} \, ds = C_0 + \log(|x|)^{-1} \) for \( |x| \leq 1 \). Therefore we obtain the desired estimate for \( G_k^{\alpha}(x, \lambda) \).

It remains to show that \( G_k^{\alpha}(\cdot, \lambda) \in C^\infty(\mathbb{R}^n \setminus \{0\}) \). Let \( N \) be a positive integer satisfying \( m(k + l) > n + |\alpha| + N \). In view of \( \xi^{\alpha+\beta}(a(\xi) - \lambda)_{-k-l} \in L_1(\mathbb{R}^n) \) for \( |\beta| \leq N \) we know that \( G_{k+l}^{\alpha}(\cdot, \lambda) \in C^N(\mathbb{R}^n) \) and \( \partial^\beta G_{k+l}^{\alpha}(x, \lambda) = \hat{i}^{\beta} G_{k+l}^{\alpha+\beta}(x, \lambda) \) for \( |\beta| \leq N \). By the evaluation for \( G_{k+l}^{\alpha+\beta}(x, \lambda) \) we may carry out the differentiation under the integral sign \( N \) times in (7) whenever \( x \neq 0 \). Therefore \( G_k^{\alpha}(\cdot, \lambda) \in C^N(\mathbb{R}^n \setminus \{0\}) \). Since we can take \( l \) and hence \( N \) arbitrarily large, we conclude that \( G_k^{\alpha}(\cdot, \lambda) \in C^\infty(\mathbb{R} \setminus \{0\}) \). This completes the proof of Theorem 1.1. \( \square \)
Remark 1.4. Loosely speaking, the proof of Theorem 1.1 is based on the following two facts. Firstly, since the integral $c_{k,l} \int_{L(\lambda)} (\mu - \lambda)^{|l-1}(a(\xi) - \mu)^{-k-1} d\mu$ converges to $(a(\xi) - \mu)^{-k}$ in $S'(\mathbb{R}^n)$, the inverse Fourier transformation yields (7) as an identity in $S'(\mathbb{R}^n)$. Secondly, by the exponential decay estimate for $G_{k+l}(x, \mu)$ we know that (7) holds as an identity in $L_1(\mathbb{R}^n)$.

2 Elliptic operators with variable coefficients

Our method of using Lemma 1.2 also works for elliptic operators with variable coefficients defined on a domain. We here focus on the elliptic operator in non-divergence form. The divergence-form operator can be handled in a similar manner. Let us consider a strongly elliptic operator

$$a(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

and a normal set $\{b_j(x, D)\}_{j=1}^{m/2}$ of boundary operators in a domain $\Omega$ of $\mathbb{R}^n$. For $1 < p < \infty$ we define the operator $A$ in $L_p(\Omega)$ by

$$D(A) = \{u \in W_p^m(\Omega) : b_j(x, D)u|_{\partial\Omega} = 0 \text{ for } j = 1, \ldots, m/2\}$$

and $Au = a(x, D)u$ for $u \in D(A)$, where $W_p^m(\Omega)$ denotes the $L_p$-based Sobolev space of order $m$.

**Theorem 2.1.** Let $\omega \in (0, \pi/2)$. If $a(x, D)$, $\{b_j(x, D)\}_{j=1}^{m/2}$ and $\Omega$ satisfy certain conditions (see [12, Chapter 5] for the precise description of the conditions), then there exists $R > 0$ such that the resolvent $(A - \lambda)^{-1}$ exists for $\lambda \in \Lambda_\omega$ with $|\lambda| \geq R$, and that the kernel $G_\lambda(x, y)$ of $(A - \lambda)^{-1}$ satisfies the exponential decay estimate

$$|G_\lambda(x, y)| \leq C|\lambda|^{-1+n/m} \Phi(|\lambda|^{1/m}(x - y), m - n, c)$$

with some positive constants $C$ and $c$ independent of $x$, $y$ and $\lambda$.

**Remark 2.2.** See [1, 7, 8] for the theorem corresponding to elliptic operators in divergence form.

The proof of Theorem 2.1 is given by Tanabe [12] (see also [11]) and it can be summarized as follows. Using the inequalities

$$\|D^\alpha (A - \lambda)^{-1}\|_{L_p(\Omega) \to L_p(\Omega)} \leq C|\lambda|^{-1+|\alpha|/m}$$

for $|\alpha| \leq m$, and the similar inequalities for the adjoint of $A$, we know by the Sobolev embedding theorem that for a positive integer $k$ with $mk > n$ the
operator \((A - \lambda)^{-k}\) has a continuous kernel \(G^k_\lambda(x, y)\) satisfying \(|G^k_\lambda(x, y)| \leq C|\lambda|^{-k+n/m}\). Considering the operator \(e^{x\eta}a(x, D)(e^{-x\eta}u)\) with suitable \(\eta \in \mathbb{R}^n\), we get
\[
|G^k_\lambda(x, y)| \leq C|\lambda|^{-k+n/m} \exp(-c|\lambda|^{1/m}|x - y|). \tag{10}
\]
We derive the Gaussian estimate for the kernel of the semigroup \(\exp(-tA)\) via the formula
\[
\exp(-tA) = \frac{(k-1)!t^{1-k}}{2\pi i} \int_\Gamma e^{-t\lambda}(A - \lambda)^{-k} d\lambda, \tag{11}
\]
where \(\Gamma\) is a suitable contour in \(\mathbb{C}\). We finally obtain (8) via the formula
\[
(A - \lambda)^{-1} = \int_0^\infty e^{t\lambda} \exp(-tA) dt \tag{12}
\]
for \(\lambda < 0\) and its analytic continuation. Thus we must go back and forth between resolvents and semigroups.

In the proof of Theorem 2.1 we can take a shortcut, if we use the identity
\[
(A - \lambda)^{-1} = c_k \int_{L(\lambda)} (\mu - \lambda)^{k-2}(A - \mu)^{-k} d\mu \tag{13}
\]
with \(c_k = (-1)^{k-1}(k-1)\) for an integer \(k \geq 2\), which corresponds to the identity in Lemma 1.2. Identity (13) can be obtained in the same way as Lemma 1.2 by noting that
\[
\|(A - \lambda)^{-k}\|_{L_p(\Omega) \to L_q(\Omega)} \leq C|\lambda|^{-k} \tag{13}
\]
by (9), and that the integral in (13) converges in the Banach space of all bounded linear operators in \(L_p(\Omega)\).

Let us evaluate the kernel of \((A - \lambda)^{-1}\) directly from that of \((A - \lambda)^{-k}\) without passing through the kernel of semigroup. For \(f \in L_p(\Omega)\) and \(g \in L_q(\Omega)\) with \(p^{-1} + q^{-1} = 1\) we set \(\langle f, g \rangle_\Omega = \int_\Omega f(x)g(x) dx\), \(\|f\|_p = \|f\|_{L_p(\Omega)}\) and \(\|g\|_q = \|g\|_{L_q(\Omega)}\). Using (10) and (13) with an integer \(k\) satisfying \(k \geq 2\) and \(mk > n\), we have
\[
\langle (A - \lambda)^{-1}f, g \rangle_\Omega = c_k \int_{L(\lambda)} (\mu - \lambda)^{k-2} \langle (A - \mu)^{-k}f, g \rangle_\Omega d\mu
\]
\[
= c_k \int_{L(\lambda)} \left[ \int_\Omega \left( \int_\Omega M(x, y, \mu) dy \right) dx \right] d\mu,
\]
where
\[
M(x, y, \mu) = (\mu - \lambda)^{k-2} C^k_\mu(x, y) f(y)g(x).
\]
Since (10) and Young’s inequality for convolutions give
\[
\int_{\mathbb{L}(\lambda)} \left[ \int_{\Omega} \left( \int_{\Omega} M(x, y, \mu) \, dy \right) \, dx \right] \, d\mu \\
\leq C \|f\|_p \|g\|_q \int_{\mathbb{L}(\lambda)} |\mu - \lambda|^{k-2} |\mu|^{-k+n/m} \| \exp(-c|\mu|^{1/m} \cdot |\cdot|) \|_{L_1(\mathbb{R}^n)} \, |d\mu| \\
\leq C c^{-n} \|f\|_p \|g\|_q \int_{|\lambda|}^{\infty} t^{-2} \, dt < \infty,
\]
we conclude by Fubini’s theorem that \((A - \lambda)^{-1}\) is an integral operator with kernel \(G_\lambda(x, y)\) given by
\[
G_\lambda(x, y) = c_k \int_{\mathbb{L}(\lambda)} (\mu - \lambda)^{k-2} G_\mu^k(x, y) \, d\mu. \tag{14}
\]
For any \(\epsilon > 0\) the last integral converges uniformly with respect to \((x, y)\) for \(|x - y| \geq \epsilon\) by (10). Hence \(G_\lambda(x, y)\) is continuous off-diagonal. Estimate (8) follows from (10) and (14) in the same way as in the proof of Theorem 1.1.

Thus we can prove Theorem 2.1, using (13) instead of (11) and (12).

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**References**


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