Multilevel Distance Labeling for Generalized Petersen $P(4k + 2, 2)$ related Graphs

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Abstract

A multilevel distance labeling of the graph $G$ is a function $f = (V(G), E(G))$ on $V(G)$ of $G$ into $\mathbb{N} \cup \{0\}$ so that $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$ for all $u, v \in V(G)$. The smallest span taken over all multilevel distance labeling of $G$ is radio number $rn(G)$ of $G$. In this paper, we completely determine the radio number $rn(G)$ of $G$ where $G$ is the graph obtained from generalized Petersen graph $P(n, 2)$, where $n = 4k + 2$ by adding new edges $u_iv_{i+1}$ and $u_{i+1}v_i$ i.e. $V(G) = V(P(4k + 2, 2))$ and $E(G) = E(P(4k + 2, 2)) \cup \{u_iv_{i+1}, u_{i+1}v_i : 1 \leq i \leq 4k + 2\}$.

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1 Introduction

In 19th century, the term graph labeling was introduced. Hale [3] in 1980, presented the idea for radio frequency assignment problems. Later in 2001, Chartrand et al. [1] applied this idea for assignment of channels to FM radio station. These assignment have been made on the fact that frequencies need to be assigned to the channels such that there is smallest disturbance. The geographically closed radio stations should be assigned different frequencies to avoid disturbance.

In graph theory, to model this problem, we construct a graph such that each station is represented by a vertex, and there is an edge between two vertices when there corresponding stations are close. Different authors determine the radio number for different families of graphs. A complete survey on multi-level distance labelings of graphs are given in [2]. The radio number for path and cycles was determined by [6]. The radio number for the square of paths was investigated by Liu and Xie [5] they also discussed the problem for the square of a cycle [4]. Radio Number for generalized prism graph was studied in [7]. A generalized gear graph was discussed in [8], where lower bound is given. M. M. Rivera et al. computed the radio number for cartesian product of cycles in [9]. Radio labeling for some cycle related graphs are studied by Vadiya and Vihol in [10].

Definition 1.1. Let $G$ be a connected graph, the distance $d(u, v)$ between any pair of vertices $u, v$ is the length of the shortest path between them.

Definition 1.2. The diameter of a graph is denoted by $\text{diam}(G)$ and defined as the maximum distance between any two vertices, i.e., $\text{diam}(G) = \max\{d(u, v); u, v \in G\}$.

Definition 1.3. A radio labeling which is also known as multilevel distance labeling of $G$ is a function $f : V(G) \to \mathbb{N} \cup \{0\}$ such that the inequality $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$ holds for any pair of distinct vertices $u, v$. The span of $f$ is the difference of the largest and the smallest channels used, $\max_{u,v \in V(G)}\{f(v) - f(u)\}$. The radio number of $G$ is denoted by $rn(G)$ and is defined as the minimum span of radio labeling of $G$.

Definition 1.4. The generalized Petersen graph $P(n, m)$ $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$ has a vertex set

$$V(P(n, m)) = \{u_i, v_i : i = 1, 2, \ldots n\}$$
and an edge set

\[ E(P(n, m)) = \{u_i u_{i+1}, v_i v_{i+m}, u_i v_i : \text{with indices taken modulo } n\}. \]

A graph \( G \) is obtained from generalized Petersen graph \( P(n, 2) \), where \( n = 4k + 2 \) by adding new edges \( u_i v_{i+1} \) and \( u_{i+1} v_i \), i.e., \( V(G) = V(P(4k + 2, 2)) \) and \( E(G) = E(P(4k + 2, 2)) \cup \{u_i v_{i+1}, u_{i+1} v_i : 1 \leq i \leq 4k + 2\} \). Note that, \( \text{diam}(G) = k + 2 \). For our purpose, we call the cycle induced by \( \{v_i : 1 \leq i \leq 4k + 2\} \), the inner cycle and the cycle induced by \( \{u_i : 1 \leq i \leq 4k + 2\} \), the outer cycle. The main theorems of this paper is:

**Theorem 1.5.** For graph \( G \) with order \( 8k + 4, k \geq 3 \)

\[ \text{rn}(G) = \begin{cases} 
\frac{4k^2 + 21k + 8}{2}, & \text{if } k \text{ is even;} \\
\frac{4k^2 + 21k + 7}{2}, & \text{if } k \text{ is odd.}
\end{cases} \]

2 A Lower Bound for \( G \)

In this Section, first of all we studied the lower bound for \( G \).

**Lemma 2.1.** (a) For each vertex \( u_1 \) on the outer cycle there is exactly one vertex \( u_{2k+2} \) at a distance diameter of \( G \).

(b) For each vertex \( v_1 \) on the inner cycles there is exactly one vertex \( v_{2k+2} \) at a distance diameter of \( G \).

**Proof.** (a) We show that \( d(u_1, u_{2k+2}) = k + 2 \). Since there are \( 4k + 2 \) vertices on the outer cycle of \( G \), so equal number of vertices on the left half and right half of the cycle. The path from \( u_1 \) to \( u_{2k+2} \) is of length \( k + 2 \) as

\[ u_1 \to v_{2(1)} \to v_{2(2)} \to v_{2(3)} \to \ldots \to v_{2(k+1)} \to u_{2k+2}. \]

(b) \( d(v_1, v_{2k+2}) = k + 2 \), as

\[ v_1 \to v_{2(1)+1} \to v_{2(2)+1} \to \ldots \to v_{2(k)+1} \to u_{2k+1} \to v_{2k+2}. \]

This completes the proof.

**Lemma 2.2.** (a) For any three vertices \( x, y, z \) on the outer cycle of \( G \),

\[ d(x, y) + d(y, z) + d(z, x) \leq 2\text{diam}(G) + 1. \]

(b) For any three vertices \( x, y, z \) on the inner cycles of \( G \),

\[ d(x, y) + d(y, z) + d(z, x) \leq 2\text{diam}(G). \]

(c) If two vertices are on the outer cycle and one vertex on the inner cycles of \( G \), then

\[ d(x, y) + d(y, z) + d(z, x) \leq 2\text{diam}(G). \]
Proof. (a) Using Lemma 2.1(a), \( d(u_1, u_{2k+2}) = k + 2 = \text{diam}(G) \).

Case 1. (even value of \( k \)) \( d(u_{2k+2}, u_{3k+4}) = \frac{k+4}{2} \) and a path of length \( \frac{k+4}{2} \) between \( u_{2k+2} \) to \( u_{3k+4} \) is

\[
u_{2k+2} \rightarrow v_{2k+3} \rightarrow v_{2(k+1)+3} \rightarrow v_{2(k+2)+3} \rightarrow \cdots \rightarrow v_{2(k+\frac{k}{2})+3}
\]

\[= v_{3k+3} \rightarrow u_{3k+4}.
\]

Also, \( d(u_{3k+4}, u_1) = \frac{k+2}{2} \) because

\[
u_{3k+4} \rightarrow v_{3k+5} \rightarrow v_{(3k+5)+1.2} \rightarrow v_{(3k+5)+2.2} \rightarrow \cdots \rightarrow v_{(3k+5)+(\frac{k-3}{2}).2}
\]

\[= v_{4k+3} = v_1 \rightarrow u_1.
\]

Therefore,

\[
d(u_1, u_{2k+2}) + d(u_{2k+2}, u_{3k+4}) + d(u_{3k+4}, u_1)
\]

\[= (k + 2) + \frac{k + 4}{2} + \frac{k + 2}{2} = 2 \text{diam}(G) + 1.
\]

Case 2. (odd value of \( k \)) \( d(u_{2k+2}, u_{3k+3}) = \frac{k+3}{2} \) and a path of length \( \frac{k+3}{2} \) between \( u_{2k+2} \) to \( u_{3k+3} \) is

\[
u_{2k+2} \rightarrow v_{2k+3} \rightarrow v_{2k+3+1.2} \rightarrow v_{2k+3+2.2} \rightarrow \cdots \rightarrow v_{2k+3+(\frac{k-1}{2}).2}
\]

\[= v_{3k+2} \rightarrow u_{3k+3}.
\]

Also, \( d(u_{3k+3}, u_1) = \frac{k+3}{2} \) because

\[
u_{3k+3} \rightarrow v_{3k+4} \rightarrow v_{(3k+4)+1.2} \rightarrow v_{(3k+4)+2.2} \rightarrow \cdots \rightarrow v_{(3k+4)+(\frac{k-1}{2}).2}
\]

\[= v_{4k+3} = v_1 \rightarrow u_1.
\]

Therefore,

\[
d(u_1, u_{2k+2}) + d(u_{2k+2}, u_{3k+3}) + d(u_{3k+3}, u_1)
\]

\[= (k + 2) + \frac{k + 3}{2} + \frac{k + 3}{2} = 2 \text{diam}(G) + 1.
\]

So if \( x, y, z \) are three vertices on the outer cycle of \( G \), then

\[
d(x, y) + d(y, z) + d(z, x) \leq 2 \text{diam}(G) + 1.
\]

(b) By Lemma 2.1(b), \( d(v_1, v_{2k+2}) = k + 2 = \text{diam}(G) \).

Case 1. (\( k \) is even) \( d(v_{2k+2}, v_{3k+4}) = \frac{k+2}{2} \) and a path of length \( \frac{k+2}{2} \) between \( v_{2k+2} \) to \( v_{3k+4} \) is

\[
v_{2k+2} \rightarrow v_{2k+2+1.2} \rightarrow v_{2k+2+2.2} \rightarrow \cdots \rightarrow v_{2k+2+(\frac{k+2}{2}).2} = v_{3k+4}.
\]

Also, \( d(v_{3k+4}, v_1) = \frac{k+2}{2} \) because

\[
v_{3k+4} \rightarrow v_{3k+4+1.2} \rightarrow v_{3k+4+2.2} \rightarrow \cdots \rightarrow v_{3k+4+(\frac{k+2}{2}).2} = v_{4k+2} \rightarrow u_1 \rightarrow v_1.
\]
Therefore,
\[ d(v_1, v_{2k+2}) + d(v_{2k+2}, v_{3k+4}) + d(v_{3k+4}, v_1) \]
\[ = (k + 2) + \frac{k + 2}{2} + \frac{k + 2}{2} = 2\text{diam}(G). \]

Case 2. \((k \text{ is odd})\) \(d(v_{2k+2}, v_{3k+3}) = \frac{k+1}{2}\) and a path of length \(\frac{k+1}{2}\) between \(v_{2k+2}\) to \(v_{3k+3}\) is
\[ v_{2k+2} \rightarrow v_{2k+2+1.2} \rightarrow v_{2k+2+2.2} \rightarrow \ldots \rightarrow v_{2k+2+(\frac{k+1}{2}).2} = v_{3k+3}. \]

Also, \(d(v_{3k+3}, v_1) = \frac{k+3}{2}\) because
\[ v_{3k+3} \rightarrow v_{3k+3+1.2} \rightarrow v_{3k+3+2.2} \rightarrow \ldots \rightarrow v_{3k+3+(\frac{k+1}{2}).2} = v_{4k+2} \rightarrow u_1 \rightarrow v_1. \]

Therefore,
\[ d(v_1, v_{2k+2}) + d(v_{2k+2}, v_{3k+3}) + d(v_{3k+3}, v_1) \]
\[ = (k + 2) + \frac{k + 1}{2} + \frac{k + 3}{2} = 2\text{diam}(G). \]

Thus if \(x, y, z\) are three vertices on the inner cycles of \(G\), then
\[ d(x, y) + d(y, z) + d(z, x) \leq 2\text{diam}(G). \]

(c) By Lemma 2.1(a), \(d(u_1, u_{2k+2}) = k + 2 = \text{diam}(G)\). For each vertex \(v_1\) on the inner cycles there is only one vertex \(u_{2k+2}\) on the outer cycle at a distance \(\text{diam}(G) - 1\), i.e., \(d(v_1, u_{2k+2}) = \text{diam}(G) - 1\). Therefore \(d(u_1, u_{2k+2}) + d(u_{2k+2}, v_1) + d(v_1, u_1) = 2\text{diam}(G)\).

Thus if \(x, y, z\) are three vertices with two vertices on the outer and one vertex on the inner cycles of \(G\), then
\[ d(x, y) + d(y, z) + d(z, x) \leq 2\text{diam}(G). \]

This completes the proof. \(\square\)

**Lemma 2.3.** Let \(f\) be a radio labeling of graph \(G\) with vertices \(8k + 4, k \geq 3\).

(a) Suppose that \(\{x_i : 1 \leq i \leq 4k + 2\}\) be the set of vertices on the outer cycle with label \(f(x_i) < f(x_{i+1})\). Then
\[ f(x_{i+2}) - f(x_i) \begin{cases} \frac{k+4}{2}, & \text{if } k \text{ is even;} \\ \frac{k+5}{2}, & \text{if } k \text{ is odd.} \end{cases} \]

(b) Suppose that \(\{y_i : 1 \leq i \leq 4k + 2\}\) be the set of vertices on the inner cycles with label \(f(y_i) < f(y_{i+1})\). Then
\[ f(y_{i+2}) - f(y_i) \begin{cases} \frac{k+5}{2}, & \text{if } k \text{ is odd;} \\ \frac{k+6}{2}, & \text{if } k \text{ is even.} \end{cases} \]
Proof. (a) Apply the radio condition to each pair in the vertex set \(\{x_i, x_{i+1}, x_{i+2}\}\), we get
\[
\begin{align*}
f(x_{i+1}) - f(x_i) &\geq \text{diam}(G) - d(x_{i+1}, x_i) + 1, \\
f(x_{i+2}) - f(x_{i+1}) &\geq \text{diam}(G) - d(x_{i+2}, x_{i+1}) + 1
\end{align*}
\]
and
\[
f(x_{i+2}) - f(x_i) \geq \text{diam}(G) - d(x_{i+2}, x_i) + 1.
\]
Sum up the above three inequalities
\[
2[f(x_{i+2}) - f(x_i)] \geq 3 + 3\text{diam}(G) - 2\text{diam}(G) = \text{diam}(G) + 3.
\]
Using Lemma 2.2(a)
\[
2[f(x_{i+2}) - f(x_i)] \geq 3 + 3\text{diam}(G) - 2\text{diam}(G) = \text{diam}(G) + 3.
\]
Since \(\text{diam}(G) = k + 2\), thus
\[
f(x_{i+2}) - f(x_i) \geq \begin{cases} 
\frac{k+4}{2}, & \text{if } k \text{ is even;} \\
\frac{k+5}{2}, & \text{if } k \text{ is odd.}
\end{cases}
\]
(b) Now let \(\{y_i, y_{i+1}, y_{i+2}\}\) be any set of three vertices on the inner cycles of \(G\). Apply radio condition to each pair in the above manners and use Lemma 2.2(b), we obtain
\[
2[f(y_{i+2}) - f(y_i)] \geq 3 + 3\text{diam}(G) - 2\text{diam}(G) = \text{diam}(G) + 3.
\]
As \(\text{diam}(G) = k + 2\), so
\[
f(y_{i+2}) - f(y_i) \geq \begin{cases} 
\frac{k+5}{2}, & \text{if } k \text{ is odd;} \\
\frac{k+6}{2}, & \text{if } k \text{ is even.}
\end{cases}
\]
This completes the proof. \(\square\)

Theorem 2.4. For a graph \(G\) with order \(8k + 4\), \(k \geq 3\)
\[
\text{rn}(G) \geq \begin{cases} 
\frac{4k^2 + 21k + 8}{2}, & \text{if } k \text{ is even;} \\
\frac{4k^2 + 21k + 7}{2}, & \text{if } k \text{ is odd.}
\end{cases}
\]

Proof. The graph have \(8k+4\) vertices. Divide set of vertices into two subsets \(\{u_1, u_2, u_3, \ldots, u_{4k+2}\}\) and \(\{v_1, v_2, v_3, \ldots, v_{4k+2}\}\). Let \(f\) be a distance labeling for \(G\). We order the vertices of \(G\) on the outer cycle by \(x_1, x_2, x_3, \ldots, x_{4k+2}\) with \(f(x_i) < f(x_{i+1})\) and vertices on the inner cycles by \(y_1, y_2, y_3, \ldots, y_{4k+2}\) with \(f(y_i) < f(y_{i+1})\). Then \(\text{diam}(G) = k + 2\). For \(i = 1, 2, 3, \ldots, 4k + 1\), set
\[d_i = d(x_i, x_{i+1})\] and \(f_i = f(x_{i+1}) - f(x_i)\). By definition \(f_i \geq \text{diam}(G) - d_i + 1\) for all \(i\). By Lemma 2.3(a), the span of a distance labeling of \(G\) for the vertices on the outer cycle is

\[
f(x_{4k+2}) = \sum_{i=1}^{4k+1} f_i = f_1 + f_2 + f_3 + \ldots + f_{4k} + f_{4k+1}
\]

\[= [f(x_2) - f(x_1)] + [f(x_3) - f(x_2)] + [f(x_4) - f(x_3)] + \ldots + [f(x_{4k+1}) - f(x_{4k})] + [f(x_{4k+2}) - f(x_{4k+1})]
\]

\[= (f_1 + f_2) + (f_3 + f_4) + (f_4 + f_5) + \ldots + (f_{4k-1} + f_{4k}) + f_{4k+1}
\]

\[= \sum_{i=1}^{4k} (f_{2i-1} + f_{2i}) + f_{4k+1},\]

\[f(x_{4k+2}) \geq \begin{cases} \frac{4k+2-2}{2}(k+5) + 1, & \text{if } k \text{ is even;} \\ \frac{4k+2-2}{2}(k+5) + 1, & \text{if } k \text{ is odd;} \end{cases}\]

\[f(x_{4k+2}) \geq \begin{cases} k^2 + 4k + 1, & \text{if } k \text{ is even;} \\ k^2 + 5k + 1, & \text{if } k \text{ is odd.} \end{cases}\]

Apply Lemma 2.2 and Lemma 2.3(b) to the vertices \(x_{4k+1}, x_{4k+2}, y_1\) such that \(f(x_{4k+1}) < f(x_{4k+2}) < f(y_1)\), then

\[f(y_1) - f(x_{4k+1}) \geq \begin{cases} \frac{k+6}{2}, & \text{if } k \text{ is even;} \\ \frac{k+5}{2}, & \text{if } k \text{ is odd;} \end{cases}\]

\[f(y_1) \geq \begin{cases} f(x_{4k+1}) + \frac{k+6}{2} = \frac{2k^2+9k+6}{2}, & \text{if } k \text{ is even;} \\ f(x_{4k+1}) + \frac{k+5}{2} = \frac{2k^2+11k+5}{2}, & \text{if } k \text{ is odd.} \end{cases}\]

By Lemma 2.3(b), the span of distance labeling of \(G\) for the vertices on the inner cycles is

\[
f(y_{4k+2}) - f(y_1)
\]

\[= \sum_{i=1}^{4k+1} f_i = (f_1 + f_2) + (f_2 + f_3) + \ldots + (f_{4k-1} + f_{4k}) + f_{4k+1}
\]

\[= \sum_{i=1}^{4k} (f_{2i-1} + f_{2i}) + f_{4k+1},\]

\[f(y_{4k+2}) - f(y_1) \geq \begin{cases} \frac{4k+2-2}{2}(k+6) + 1, & \text{if } k \text{ is even;} \\ \frac{4k+2-2}{2}(k+5) + 1, & \text{if } k \text{ is odd;} \end{cases}\]
\[ f(y_{4k+2}) \geq \begin{cases} k^2 + 6k + 2 + f(y_1), & \text{if } k \text{ is even;} \\ k^2 + 5k + 1 + f(y_1), & \text{if } k \text{ is odd;} \end{cases} \]

\[ f(y_{4k+2}) \geq \begin{cases} \frac{4k^2 + 21k + 8}{2}, & \text{if } k \text{ is even;} \\ \frac{4k^2 + 21k + 7}{2}, & \text{if } k \text{ is odd.} \end{cases} \]

This completes the proof. \(\square\)

\textbf{Figure 1:} Ordinary labeling and radio labeling for \(G\), when \(k\) is even

\section{Upper Bound for \(G\)}

To complete the proof of Theorem 1.5, it remains to find distance labeling for \(G\) with span equal to the desired numbers. The labeling is generated by

pair of three sequences, the distance gap sequences

\[ D = (d_1, d_2, d_3, \ldots, d_{4k+1}), \quad D' = (d'_1, d'_2, d'_3, \ldots, d'_{4k+1}), \]

the color gap sequence

\[ F = (f_1, f_2, f_3, \ldots, f_{4k+1}), \quad F' = (f'_1, f'_2, f'_3, \ldots, f'_{4k+1}), \]

and the vertex gap sequences

\[ T = (t_1, t_2, t_3, \ldots, t_{4k+1}), \quad T' = (t'_1, t'_2, t'_3, \ldots, t'_{4k+1}). \]

Case 1. (\(k\) is even) The distance gap sequences \(D\) and \(D'\) are given by

\[ d_i = d'_i = \begin{cases} k + 2, & \text{if } i \text{ is odd;} \\ \frac{k+1}{2}, & \text{if } i \text{ is even.} \end{cases} \]

For each \(i\), We have \(d(x_i, x_{i+1}) = d_i\), \(d(y_i, y_{i+1}) = d'_i\) and

\[ d' = d(x_{4k+2}, y_1) = \frac{k + 2}{2}. \]
The color gap sequences $F$ and $F'$ are given by:

$$f_i = \begin{cases} 
1, & \text{if } i \text{ is odd}; \\
\frac{k+2}{2}, & \text{if } i \text{ is even}
\end{cases}$$

and $f' = \frac{k+4}{2}$. The vertex gap sequences $T$ and $T'$ are given by:

$$t_i = \begin{cases} 
2k, & \text{if } i \text{ is odd}; \\
k + 1, & \text{if } i \text{ is even}
\end{cases}$$

$$t'_i = \begin{cases} 
2k, & \text{if } i \text{ is odd}; \\
k, & \text{if } i \text{ is even}
\end{cases}$$

$t_i$ denotes number of vertices between $x_i$ and $x_{i+1}$ on the outer cycle and $t'_i$ denotes number of vertices between $y_i$ and $y_{i+1}$ on the inner cycles. Let $\pi, \pi' : \{1, 2, 3, \ldots, 4k + 2\} \rightarrow \{1, 2, 3, \ldots, 4k + 2\}$ be defined by $\pi(1) = 1$ and

$$\pi'(1) = \begin{cases} 
4k + 2, & \text{if } k = 6s \text{ and } k = 6s + 2 \text{ where } s \geq 1; \\
4k - 1, & \text{if } k = 6s - 2 \text{ where } s \geq 1,
\end{cases}$$

$$\pi(i + 1) = \pi(i) + t_i + 1 \pmod{4k + 2},$$

$$\pi'(i + 1) = \pi'(i) + t'_i + 1 \pmod{4k + 2}.$$
If $\pi(2i) = \pi(2i' - 1)$, then we get

\[(i - i')(3k + 3) \equiv -(2k + 1) \pmod{4k + 2},\]
\[(i - i')(2k + 4) \equiv 0 \pmod{4k + 2}.

As the greatest common divisor $(4k + 2, 2k + 4) = 2$, this means $i' - i \equiv 0 \pmod{2k + 1}$, which is a contradiction to the fact $0 < i - i' < 2k + 1$. Now, show that $\pi'$ is a permutation. Since greatest common divisor $(4k + 2, k) = 2$ and $3k + 2 \equiv -k \pmod{4k + 2}$. It follows that $(3k + 2)(i - i') \equiv k(i' - i) \neq 0 \pmod{4k + 2}$ for $0 < i - i' < 2k + 1$. This implies that $\pi'(2i) \neq \pi'(2i')$ for $i \neq i'$.

Similarly $\pi'(2i - 1) \neq \pi'(2i' - 1)$ for $i \neq i'$.

If $\pi'(2i) = \pi'(2i' - 1)$, then

\[(3k + 2)(i - i') \equiv -(2k + 1) \pmod{4k + 2},\]
\[(2k + 2)(i' - i) \equiv 0 \pmod{4k + 2}.

Since the greatest common divisor $(4k + 2, 2k + 2) = 2$, it follows that $i' - i \equiv 0 \pmod{2k + 1}$, which is not possible.

The span of $f$ is equal to

\[f_1 + f_2 + f_3 + \ldots + f_{4k} + f_{4k+1} + f' + f'_1 + f'_2 + f'_3 + \ldots + f'_{4k} + f'_{4k+1} = \left[ (f_1 + f_3 + f_5 + \ldots + f_{4k+1}) \right] + \left[ (f_2 + f_4 + f_6 + \ldots + f_{4k}) \right] + f' + \left[ (f'_1 + f'_3 + f'_5 + \ldots + f'_{4k+1}) \right] + \left[ (f'_2 + f'_4 + f'_6 + \ldots + f'_{4k}) \right] = \frac{4k + 2}{2} \left( \frac{k + 2}{2} \right) + \frac{k + 4}{2} + \frac{4k + 2}{2} \left( 1 + \frac{4k}{2} \right) \left( \frac{k + 4}{2} \right) = \frac{4k^2 + 21k + 8}{2}.

Case 2. ($k$ is odd) The distance gap sequences are given by

\[d_i = d'_i = \begin{cases} k + 2, & \text{if } i \text{ is odd;} \\ \frac{k + 3}{2}, & \text{if } i \text{ is even.} \end{cases}

Where $d'_i = d(y_i, y_{i+1})$ for $i = 1, 2, 3, \ldots, 4k + 1$ and

\[d' = d(x_{4k+2}, y_1) = \frac{k + 3}{2}.

The color gap sequences $F$ and $F'$ are given by:

\[f_i = f'_i = \begin{cases} 1, & \text{if } i \text{ is odd;} \\ \frac{k + 3}{2}, & \text{if } i \text{ is even.} \end{cases}
and $f' = \frac{k+3}{2}$.

The vertex gap sequences $T$ and $T'$ are given by

$$
t_i = \begin{cases} 
2k, & \text{if } i \text{ is odd; } \\
k, & \text{if } i \text{ is even, }
\end{cases}
\quad t_i' = \begin{cases} 
2k, & \text{if } i \text{ is odd; } \\
k - 1, & \text{if } i \text{ is even. }
\end{cases}
$$

Let $\theta, \theta': \{1, 2, 3, \ldots, 4k + 2\} \to \{1, 2, 3, \ldots, 4k + 2\}$ be defined as $\theta(1) = 1$, $\theta'(1) = 4k + 2$,

$$
\theta(i + 1) = \theta(i) + t_i + 1,
\quad \theta'(i + 1) = \theta'(i) + t_i' + 1.
$$

Then for $i = 1, 2, 3, \ldots, 2k + 1,$

$$
\theta(2i - 1) = 3k(i - 1) + 2i - 1 \pmod{4k + 2},
\quad \theta(2i) = 3i - 1)k + 2i \pmod{4k + 2},
$$

and

$$
\theta'(2i - 1) = 4k + 2 + (3k + 1)(i - 1) \pmod{4k + 2},
\quad \theta'(2i) = 4k + 2 + (3i - 1)k + i \pmod{4k + 2}.
$$

We prove that $\theta$ and $\theta'$ are permutations. Note that the greatest common divisor $(4k + 2, k) = 1$, and $(3k + 2) \equiv -k \pmod{4k + 2}$. This shows that $(3k + 2)(i - i') \equiv k(i' - i) \not\equiv 0 \pmod{4k + 2}$ when $0 < i - i' < 2k + 1$. Thus $\theta(2i) \neq \theta(2i')$ for $i \neq i'$.

Similarly, $\theta(2i - 1) \neq \theta(2i' - 1)$ for $i \neq i'$.

However, if $\theta(2i) = \theta(2i' - 1)$, then

$$
(3k + 2)(i - i') \equiv (2k + 1)(i' - i) \pmod{4k + 2},
\quad (2k + 2)(i - i') \equiv 0 \pmod{4k + 2}.
$$

Since the greatest common divisor $(4k + 2, 2k + 2) = 2$, which implies that $i - i' \equiv 0 \pmod{2k + 1}$. This means $2k + 1$ divides $i - i'$, which is not possible because $0 < i - i' < 2k + 1$. Therefore $\theta$ is a permutation. Now, we show that $\theta'$ is a permutation. As the greatest common divisor $(4k + 2, k) = 1$, and $3k + 1 \equiv -k - 1 \pmod{4k + 2}$. This implies that $(3k + 1)(i - i') \equiv (k + 1)(i' - i) \not\equiv 0 \pmod{4k + 2}$ when $0 < i' - i < 2k + 1$. Thus $\theta'(2i) \neq \theta'(2i')$ for $i \neq i'$.

Similarly $\theta'(2i - 1) \neq \theta'(2i' - 1)$ for $i \neq i'$.

However, if $\theta(2i) = \theta(2i' - 1)$, then we obtain

$$
(3k + 1)(i - i') \equiv -(2k + 1)(i - i') \pmod{4k + 2},
\quad 2k(i - i') \equiv 0 \pmod{4k + 2}.
$$

As the greatest common divisor $(4k + 2, 2k) = 2$ implies that $i - i' \equiv 0 \pmod{2k + 1}$, which means $2k + 1/i - i' < 2k + 1$. So, $\theta'$ is a permutation.
The span of $f$ is equal to

\[ f_1 + f_2 + f_3 + ... + f_{4k+1} + f'_1 + f'_2 + f'_3 + ... + f'_{4k+1} = [(f_1 + f_3 + f_5 + ... + f_{4k+1})] + [(f_2 + f_4 + f_6 + ... + f_{4k})] + [f'_1 + f'_3 + f'_5 + ... + f'_{4k+1}]] + [(f'_2 + f'_4 + f'_6 + ... + f'_{4k})] \]

\[ = \frac{4k + 2}{2}(1) + \frac{4k}{2} \left( \frac{k + 3}{2} \right) + \frac{k + 3}{2} + \frac{4k + 2}{2}(1) + \frac{4k}{2} \left( \frac{k + 3}{2} \right) \]

\[ = \frac{4k^2 + 21k + 7}{2}. \]

**Figure 2:** Ordinary labeling and radio labeling for $G$, when $k$ is odd.

**References**


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