Some Inequalities for Measurable Operators

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Abstract

In this paper, Loewner-Heinz inequality, Hansen inequality and Furuta inequality for measurable operators affiliated with a given von Neumann algebras will be given.

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1 Introduction

Let $\Omega$ be an interval on the real line and let $f$ be a real continuous function on $\Omega$. Let $\mathcal{M}$ denotes a von Neumann $*$-subalgebra of algebra $B(H)$ of all bounded linear operators on Hilbert space $H$. The function $f$ is then said to be $\mathcal{M}$-monotone on $\Omega$ (or operator monotone with respect to $\mathcal{M}$ on $\Omega$) if for any pair of self-adjoint operators $A, B$ in $\mathcal{M}$ with spectra in $\Omega$,

$$A \leq B \implies f(A) \leq f(B).$$

If $\mathcal{M} = B(H)$, then $f$ is called a matrix monotone if $dimH = n < \infty$, and a operator monotone if $dimH = \infty$.

The theory of matrix and operator monotone functions was introduced and developed by K.Loewner [1] and his two students O.Dobsch and F.Kraus in 1934-1936. Such functions have a great variety of applications to many fields of both pure and applied mathematics, specially, in operator algebra theory.
and quantum information theory. Famous Loewner-Heinz inequality says that, for any \( p \in [0, 1] \) and matrices \( A, B \),

\[
0 \leq A \leq B \quad \text{implies} \quad A^p \leq B^p.
\]

In [2], Furuta gave an extension of Loewner-Heinz inequality as follows:

**Theorem 1.1.** If \( 0 \leq A \leq B \) are positive matrices of order \( n \times n \), then

\[
A^{(p+2r)/q} \leq (A^r B^p A^r)^{1/q}
\]

and

\[
B^{(p+2r)/q} \geq (B^r A^p B^r)^{1/q}
\]

for \( r \geq 0, p \geq 0, q \geq 1 \) with \( (1 + 2r)q \geq p + 2r \).

In this paper, we extend Loewner-Heinz inequality to measurable operators with respect to a von Neumann algebra [4]. Using obtained one, we prove Hansen inequality and Furuta inequality for this class of operators.

## 2 Preliminary denotations

Throughout this paper \( \mathcal{M} \) denotes a von Neumann subalgebra of algebra \( B(H) \) of all bounded linear operators on Hilbert space \( H \). The lattice of all orthogonal projections in \( \mathcal{M} \) is denoted by \( \mathcal{M}^{pr} \) and \( \mathcal{M}^{un} \) is the unitary group of \( \mathcal{M} \).

Let us recall some notions from general noncommutative integral theory developed by Segal [4]. A linear operator \( X \) in \( H \), with domain \( D(X) \), is said to be **affiliated with a von Neumann algebra** \( \mathcal{M} \), if \( UX = XU \) for all \( U \in \mathcal{M}^{un} \).

A linear subspace \( D \) of \( H \) is said to be **strongly dense** in \( H \) with respect to \( \mathcal{M} \), if there exists a sequence \( \{ P_n \} \) in \( \mathcal{M}^{pr} \) such that \( P_n \uparrow 1, P_n(H) \subset D \) and \( P_n (= 1 - P_n) \) is a finite projection for every \( n = 1, 2, \ldots \).

A linear closed operator \( X \) in the Hilbert space \( H \) is called **measurable with respect to the von Neumann algebra** \( \mathcal{M} \), if:

1. \( X \) is affiliated with \( \mathcal{M} \);
2. the domain \( D(X) \) of \( X \) is strongly dense in \( H \).

The set of all measurable operators with respect to \( \mathcal{M} \) is denoted by \( S(\mathcal{M}) \). For measurable \( A \) and \( B \), \( A^* \) and \( B^* \) are measurable. Also, the (algebraic) sum \( A + B \) and the (algebraic) product \( AB \) are closable, and their closures \( (A+B)^- \) (the strong sum) and \( (AB)^- \) (the strong product) are again measurable. If \( f \) is a Borel function which is bounded on compact subsets of spectrum \( \sigma(A) \) of \( A \), then \( f(A) \in S(\mathcal{M}) \). Furthermore, under strong sum, strong product, and
the adjoint operation, the set of all measurable operators with respect to a von Neumann algebra forms a *-algebra.

A sequence \( \{T_n\} \in S(M) \) is said to **converge nearly everywhere** (n.e.) to a measurable \( T \) in \( S(M) \), if for every positive \( \varepsilon \) there exists a sequence of projections \( P_n \) in \( M^{pr} \) such that \( P_n \uparrow 1 \) as \( n \uparrow \infty \), \( \| (T_n - T) \cdot P_n \| < \varepsilon \) \((n = 1, 2, \cdots)\) and \( 1 - P_n \) is finite.

\section{3 Results}

**Theorem 3.1 ([5][Theorem 5].)** Let \( M \) be a von Neumann algebra on Hilbert space \( H \) and function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be an operator monotone with respect to \( M \). Then \( f(A) \leq f(B) \) for any pair of positive self-adjoint operators \( A, B \) affiliated with \( M \) such that \( A \leq B \).

**Corrolary 3.1 (Loewner-Heinz inequality for measurable operators).** If \( 0 \leq A \leq B \) in \( S(M) \) and \( r \in [0, 1] \) then

\[ A^r \leq B^r. \tag{1} \]

**Proof.** For \( 0 \leq r \leq 1 \) the function \( f(t) = t^r \) \((t \geq 0)\) is operator monotone with respect to \( M \). Hence the inequality (1) follows from the Theorem 3.1. \( \square \)

**Corrolary 3.2 (Hansen’s inequality for measurable operators).** Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be an operator monotone function with respect to \( M \), \( Y \in M \), \( \|Y\| \leq 1 \) and \( X \in S(M)^+ \). Then

\[ Y^* f(X) Y \leq f(Y^* X Y). \tag{2} \]

**Proof.** Put \( X_n = X \cdot \chi([0,n]) \). It is clear that \( \{X_n\} \) is an increasing sequence of positive operators in \( M \) and converges n.e. to \( X \). Remark that \( X_n \) commute with \( X \) for every \( n \). So the convergence n.e. of the sequence \( X_n \) to \( X \) can be considered as in the commutative case. Therefore, for operator monotone function \( f \) with respect to \( M \), and so is continuous on \( \mathbb{R}^+ \), sequence \( f(X_n) \) converges n.e. to \( f(X) \). By Theorem 3.1 we also have

\[ f(X_n) \leq f(X_{n+1}) \leq \cdots \leq f(X) \quad \text{(since } X_n \leq X_{n+1} \leq \cdots \leq X). \]

Consequently, \( Y^* f(X_n) Y \) converge n.e. to \( Y^* f(X) Y \). On another hand, for every \( X_n \) by the Hansen’s inequality [3] we have

\[ Y^* f(X_n) Y \leq f(Y^* X_n Y) \leq f(Y^* X Y). \]

Tending \( n \uparrow \infty \), we obtain

\[ Y^* f(X) Y \leq f(Y^* X Y). \]

\( \square \)
The proof of the following theorem is adapted from [2].

**Theorem 3.2.** If $0 \leq A \leq B$ in $S(M)^+$, then

$$A^{(p+2r)/q} \leq (A^p B^r A^r)^{1/q}$$

(3)

and

$$B^{(p+2r)/q} \geq (B^p A^r)^{1/q}$$

(4)

for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$.

**Proof.** We first prove (3).

If $p \in [0, 1]$, then by Corollary 3.1 $A^p \leq B^p$ and hence $A^{p+2r} \leq A^r B^p A^r$. Applying Corollary 3.1 again with the power $1/q \in (0, 1]$ gives (3).

Next we consider the case $p \geq 1$. It suffices to prove

$$A^{1+2r} \leq (A^p B^p A^r)^{(1+2r)/(p+2r)}$$

(5)

for $r \geq 0, p \geq 1$, since by assumption $q \geq (p + 2r)/(1 + 2r)$, then (3) follows from this inequality via Corollary 3.1. Without loss of generality we can assume that $A$ and $B$ are invertible. In the case $0 \leq r \leq 1/2$, $0 \leq A \leq B$ implies $A^{2r} \leq B^{2r}$ by Corollary 3.1, so $A^r B^{-2r} A^r \leq 1$, namely $\|B^{-r} A^r\|_M \leq 1$. Put $q = (p + 2r)/(1 + 2r) \geq 1$. By Corollary 3.2, we have

$$(A^p B^p A^r)^{1/q} = (A^r B^{-r} B^{p+2r} B^{-r} A^r)^{1/q} \geq A^r B^{-r} B^{(p+2r)/q} B^{-r} A^r = A^r B A^r \geq A^{1+2r}.$$

So, for $0 \leq r \leq 1/2$ the last inequality is nothing but (5). Applying (5) again for $0 \leq r_1 \leq 1/2$ and $p_1 \geq 1$, we get

$$A^{(1+2r)(1+2r_1)} \leq A^{(1+2r)r_1} (A^r B^p A^r)^{p_1/q} A^{(1+2r)r_1 1/q_1},$$

for $q_1 = (p_1 + 2r_1)/(1 + 2r_1)$.

Put $p_1 = q$. Then we have

$$A^{(1+2r)(1+2r_1)} \leq A^{(1+2r)r_1 + r} B^p A^{r_1 + r} A^{1+2r} r_1 1/q_1.$$

(6)

Put $r_2 = (2r + 1)r_1 + r$. Then $q_1 = (q_1 + 2r_1)/(p_1 + 2r_1)/(1 + 2r_2)$ since $p_1 = q$ and $(1 + 2r)(1 + 2r_1) = 2r_2 + 1$. Consequently, (6) means that (5) holds for $r_2 \in [0, 3/2]$ since $r_1 \in [0, 1/2]$. Repeating above argument, we get that (5) holds for each $r \geq 0$.

Inequality (4) is easily obtained by (3) as follow. Without loss of generality, we assume that $A^{-1}$ exists. By hypothesis, $0 \leq B^{-1} \leq A^{-1}$. Then by (3), for each $r \geq 0$, $B^{-r} B^{-2r}/q \leq (B^{-r} A^{-p} B^{-r})^{1/q}$ holds for each $p$ and $q$ such that $p \geq 0, q \geq 1$, and $(1 + 2r)q \geq p + 2r$. Inequality (4) is the inverse of the last inequality. \qed
Corollary 3.3. Let $\tau$ be a faithful normal semi-finite trace on a von Neumann algebra $M$ and $r, p \geq 0$. Then for any pair of positive selfadjoint measurable operators $A \leq B \in S(M)$,

$$\tau(A^{p+2r}) \leq \tau(A^r B^p A^r)$$

and

$$\tau(B^{p+2r}) \geq \tau(B^r A^p B^r).$$

Proof. Since the function $f(t) = t^q$ is monotone on $[0, \infty)$ when $q \geq 1$. From the monotonicity of the trace and previous Furuta inequalities, we get conclusion. \qed

Remark 1. Note that inequalities in Corollary 3.3 characterize the trace property among normal weights on von Neumann algebras (see, for example, [8]).

References


[2] Furuta T. $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$. Proc. Amer. Math. Soc. 101 (1987) 85-88.


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