Negative Theorem for Monotone Bivariate Approximation

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Abstract. When we approximate a monotone function \( f \) in \( L_p[-1,1]^2 \), we wish some times that the approximating polynomial is also monotone. This approximation is accomplished quantitatively by the use of suitable two dimensional first modulus of smoothness, namely \( \omega_1(f, \delta, [-1,1]^2)_p \). Therefore a question is a raised, could we involve in our monotone approximation a higher order two dimensional modulus of smoothness. In this article we will show that this monotonicity requirement restricts very much the degree of approximation. That the polynomial can achieve the rate of \( \omega_2(f, \delta, [-1,1]^2)_p \).

1. Introduction

The first result of multi variable polynomial monotone approximation is introduced by George A. Anastassiou in [1]. He used the \( L_\infty \)-norm and raised a remark what about \( L_p \)-case. In [2], we show that for a two variable continuously differentiable real valued function of a given order, and \( L \) be a linear differential operator involving mixed derivatives in \( L_p[-1,1]^2 \), such that \( L(f) \geq 0 \). There exists a sequence of two variable polynomials \( Q_{m,n}(x,y) \) with \( L(Q_{m,n}) \geq 0 \), and
\[ f \text{ in } L_p [-1,1]^2 \text{ approximated to } Q_{m,n}(x,y) \text{ in terms of the first two variable modulus of continuity. It means we prove:} \]

**Theorem 1.1**

Let \( h_1, h_2, v_1, v_2, r, p \) be integers \( r \geq v_1 \geq h_1 \geq 0, p \geq v_2 \geq h_2 \geq 0 \) and let \( f \in L_p [-1,1]^2 \). Let \( \alpha_{ij}(x,y) = \alpha_{ij}^0 \), \( i = h_1 + 1, \ldots, v_1, j = h_2, h_2 + 1, \ldots, v_2 \) be real-valued functions defined and bounded in \( L_p [-1,1]^2 \) and assume \( \alpha_{h_1 h_2} \) is either \( \geq \alpha > 0 \) or \( \leq \beta < 0 \). Throughout \( L_p [-1,1]^2 \). Consider the operator

\[
L = \frac{\sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \partial^{i+j}}{\partial x^i \partial y^j}
\]

And assume that throughout \( L_p [-1,1]^2 \), \( L(f) \geq 0 \)

Then for integers \( m, n \) with \( m > r, n > p \) there exists a polynomial \( Q_{m,n}(x,y) \) of degree \((m,n)\) such that \( L \left( Q_{m,n}(x,y) \right) \geq 0 \) throughout \( L_p [-1,1]^2 \) and

\[
\left\| f^{(k,l)} - Q^{(k,l)}_{m,n} \right\|_p \leq \frac{P_{m,n}(L,f)}{(h_1 - k)! (h_2 - l)!} + M^{(k,l)}_{m,n}(f)
\]

all \((0,0) \leq (k,l) \leq (h_1, h_2)\). Furthermore we get

\[
\left\| f^{(k,l)} - (B_{m,n} f)^{(k,l)} \right\|_p \leq M^{(k,l)}_{m,n}(f)
\]

for all \((h_1 + 1, h_2 + 1) \leq (k,l) \leq (r,p)\). Also above inequality is true whenever \( 0 \leq k \leq h_1 \) and \( h_2 + 1 \leq l \leq p \) or \( h_1 + 1 \leq k \leq r \), \( 0 \leq l \leq h_2 \).

Here

\[
M^{(k,l)}_{m,n} \equiv M^{(k,l)}_{m,n}(f) \equiv C(k,l) \cdot \omega_1 \left( f^{(k,l)}, \left( \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}} \right) \right)_p
\]

\[
+ \max \left\{ \frac{k(k-1)}{m}, \frac{l(l-1)}{n} \right\} \cdot \left\| f^{(k,l)} \right\|_p
\]

\[
P_{m,n} \equiv P_{m,n}(L,f) \equiv \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} L_{ij} \cdot M^{(i,j)}_{m,n}
\]

where \( C \) is a positive constant depending on \( k, l \) and \( p, 0 < p < 1 \).
Let \( f \) be a real valued function defined on \([-1,1]^2\). For integers \( r, s \geq 0 \) we denote by \( f^{(r,s)} \) the differential operator of order \( (r,s) \) given by \( f^{(r,s)}(x,y) = \frac{\partial^{r+s} f(x,y)}{\partial x^r \partial y^s} \). I would like to say that we prove our result in [2] with the help of the theorem introduced by George A. Anastassiou in [1]. Also in [1], George A. Anastassiou asked: Could we involve in our inequalities a higher order modulus of smoothness? In this paper we answer this question. To complete the picture, and strengthen our theorem we provide negative theorem. Showing that our direct estimate above fail to hold in the case that the estimate in terms of \( \omega_2 \).

We would like to mention

**Definition 1.2**

Let \( f \in L_p[-1,1]^2 \), and \([-1,1]^2 = [-1,1] \times [-1,1]. The second modulus of continuity of \( f \) is defined as follows:

\[
\omega_2 \left( f, \delta, L^2 \right)_p = \sup_{|h_1| \leq \delta_1, \ |h_2| \leq \delta_2} \| f((x_1, x_2) - (h_1, h_2)) - 2f((x_1, x_2) \\
+ f((x_1, x_2) + (h_1, h_2)) \|_p.
\]

Where \( \delta = (\delta_1, \delta_2), \delta_1, \delta_2 > 0 \)

**Definition 1.3**

Let \( f \in L_p[-1,1]^2 \), The local modulus of smoothness of \( f \) is defined as:

\[
\omega_2(f, x, \delta) = \sup\{ |f((y_1, y_2) - (h_1, h_2)) - 2f(y_1, y_2) + f((y_1, y_2) \\
+ (h_1, h_2))|\}
\]

\( y_1 \pm h_1 \in [x_1 - \delta_1, x_1 + \delta_1], y_2 \pm h_2 \in [x_2 - \delta_2, x_2 + \delta_2] \)

where \( x = (x_1, x_2) \), and \( \delta = (\delta_1, \delta_2) \).
Definition 1.4

Let \( f \in L_p[-1,1]^2 \), and \( \delta_1, \delta_2 > 0 \). the \( \tau \)-modulus of smoothness of \( f \) is defined as follows:

\[
\tau_2(f, \delta, l^2)_p = \|\omega_2(f, x, \delta)\|_p
\]

where \( l^2 = [-1,1]^2 \).

2. The main Result

In this article we shall introduce our negative theorem

**Theorem 2.1**

For each \( A \geq 1 \) and \( n > \sqrt[3]{q} A \) (\( q \) is nonnegative real) there exists a continuous nondecreasing function in \( L_p[-1,1]^2 \) such that any two variable non decreasing polynomial \( p_n \) of degree \( \leq n \), necessarily satisfies

\[
\|f - p_n\|_p > A \tau_2(f, \Delta_n)_{L_p[-1,1]^2} > A \omega_2(f, \Delta_n)_{L_p[-1,1]^2} \tag{2.2}
\]

Where, \( \Delta_n(x_1) = \frac{\sqrt{1 - x_1^2}}{n} + \frac{1}{n^2} \).

**Proof:** Let

\[
f(x_1, x_2) := f_n(x_1, x_2) = \begin{cases} 
-n^{-\frac{1}{q} - 1} & (x_1, x_2) \in [-1, -\frac{1}{20n}]^2 \\
20n^{-\frac{1}{q}} x_1 & (x_1, x_2) \in (\frac{-1}{20n}, 1]^2
\end{cases}
\]

where \( q \) is non-negative real number, \( f \) is non decreasing in \( L_p[-1,1]^2 \).

\[
\omega_2(f, x, \Delta_n) = \sup \left\{ 20n^{-\frac{1}{q}} (y_1 + h_1) - 2 \left( 20n^{-\frac{1}{q}} \right) y_1 + 20n^{-\frac{1}{q}} (y_2 - h_1) \right\}
\]

\[
y_1 + h_1 \in [x_1 - \Delta_n(x_1), x_1 + \Delta_n(x_1)] , \quad y_2 + h_2 \in [x_2 - \Delta_n(x_2), x_2 + \Delta_n(x_2)]
\]

\[
\omega_2(f, x, \Delta_n) \leq 20n^{-\frac{1}{q}} \sup |h_1| , \quad h_1 \in [x_1 - \Delta_n(x_1), x_1 + \Delta_n(x_1)]
\]

\[
= 20n^{-\frac{1}{q}} |x_1 + \Delta_n(x_1)| .
\]

Then
\[\|\omega_2(f, x, \Delta_n)\|_p = \left\| 20n^{-\frac{1}{q}}(x_1 + \Delta_n(x_1)) \right\|_{L_p[-1,1]^2}\]

\[\|\omega_2(f, x, \Delta_n)\|_p = \left( \int_{-1}^{1} \int_{-1}^{1} 20n^{-\frac{1}{q}}(x_1 + \Delta_n(x_1))^p \, dx_1 \, dx_2 \right)^{\frac{1}{p}}\]

\[= 20n^{-\frac{1}{q}} \left( \frac{2}{n^2} \right)^{\frac{1}{p}}\]

Then

\[\|\omega_2(f, x, \Delta_n)\|_p \leq 2(20)^p n^{-\frac{p}{q}} \left[ (\int_{-1}^{1} |x_1|^p \, dx_1) + (\int_{-1}^{1} |\Delta_n(x_1)|^p \, dx_1) \right]\]

We have \[\sqrt{1-x^2} + \frac{1}{n^2} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}\].

Then

\[\|\omega_2(f, x, \Delta_n)\|_p \leq 2(20)^p n^{-\frac{p}{q}} \left[ (\int_{-1}^{1} |x_1|^p \, dx_1) + (\int_{-1}^{1} |\Delta_n(x_1)|^p \, dx_1) \right]\]

\[\|\omega_2(f, x, \Delta_n)\|_p \leq 2(20)^p n^{-\frac{p}{q}} \left[ 2 + \left( \frac{2}{n^2} \right)^p \right]\]

\[\leq 2(20)^p n^{-\frac{p}{q}} \left( 2 + \frac{2(2^p)}{n^p} \right)\]

\[\leq 4(20)^p n^{-\frac{p}{q}} (1 + 2^p) \leq 8(60)^p n^{-\frac{p}{q}}\]

Hence \[\tau_2(f, \Delta_n)_p \leq 60 \sqrt[20]{\frac{3}{n}}\]

If \(p_n\) is a non-decreasing polynomial not satisfies (2.2). Then

\[\|f - p_n\|_p \leq A\tau_2(f, \Delta_n)_p \leq 60 \sqrt[20]{\frac{3}{n}}\]

Since \(f\) is non-decreasing in \(L_p[-1,1]^2\) and vanishes at some point in \([-1,1]^2\).

Say \(\varepsilon \in \hat{f}\) where \(\varepsilon = (\varepsilon_1, \varepsilon_2)\) in \(\hat{f} \subset [-1,1]^2\) with measure equal to \(\frac{c}{n^2}\), therefore so is \(p_n\). This implies

\[\|p_n(x)\|_{L_p(f)} = \| \int_{\varepsilon_1}^{x_1} \int_{\varepsilon_2}^{x_2} p_n^{(1,1)} (u_1, u_2) \, du_1 \, du_2 \|_{L_p(f)}\]
\[
\int_{x_1}^{x_2} \int_{y_1}^{y_2} p_n^{(1,1)}(u_1, u_2) du_1 du_2 \right)^p dx_1 dx_2 \]

\[\|p_n(x)\|_{L_p(j)} \leq \frac{c^2}{n^2} \|p_n^{(1,1)}\|_{L_p(j)}\]

Bernstein inequality for \(\tilde{f}\) implies

\[
\left\|\left(1 - \frac{c^2}{n^4}\right) \left(1 - \frac{c^2}{n^4}\right) p_n^{(1,1)}\right\|_{L_p(j)} \leq \left\|\left(1 - \frac{c^2}{n^4}\right) p_n^{(1,1)}\right\|_{L_p(j)} \leq n^2 \|p_n\|_{L_p(j)}.
\]

Therefore,

\[
\|p_n^{(1,1)}\|_{L_p(j)} \leq \frac{n^2}{1 - \frac{c^2}{n^4}} \|p_n\|_{L_p(j)} \leq \frac{n^6}{n^4 c^2} \|p_n\|_{L_p(j)}
\]

Which is a contradiction. This completes the proof.

**Conclusions**

For a two variable continuously differentiable real valued function of a given order, and \(L\) be a linear differential operator involving mixed derivatives in \(L_p[-1,1]^2\), such that \(L(f) \geq 0\). There exists a sequence of two variable polynomials \(Q_{m,n}(x,y)\) with \(L(Q_{m,n}) \geq 0\), and \(f\) in \(L_p[-1,1]^2\) approximated to \(Q_{m,n}(x,y)\) in terms of the first two variable modulus only. This first result of multi variable polynomial monotone approximation for the \(L_p\) case.

**References**


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