Finite Codimensional Invariant Subspace and Uniform Algebra

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Abstract

Finite codimensional invariant subspaces in the abstract Hardy space defined by a unique representing measure on a uniform algebra are described.

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Let X be a compact Hausdorff space, $C(X)$ the algebra of all complex-valued continuous functions on X , and A a uniform algebra on X . Fix a nonzero complex homomorphism φ on A and a representing measure m for φ whose support is X . The abstract Hardy space $H^p = H^p(m)$, $1 \leq p \leq \infty$, determined by A is defined as the closure of A in $L^p = L^p(m)$ when p is finite and as the weak star closure of A in $L^\infty = L^\infty(m)$ when $p = \infty$. We set $A_\varphi = \{ f \in A : \varphi(f) = 0 \}$ and $H_\varphi^p = \{ f \in H^p : \int f dm = 0 \}$.

The set $\mathfrak{M}(A)$ denotes the maximal ideal space of A and X is regarded as a closed subset of $\mathfrak{M}(A)$. The set $G(\varphi)$ denotes the Gleason part of φ , that is, $G(\varphi) = \{ \psi \in \mathfrak{M}(A) : \|\psi - \varphi\| < 2 \}$. Φ is called a point derivation at φ , when Φ is a linear functional on A and $\Phi(fg) = \varphi(f)\Phi(g) + \Phi(f)\varphi(g)$ ($f, g \in A$).

For a subset M in L^p , $M_q = M \cap L^q$ and $[M]_p$ is the norm closed linear

span of M . For a subset M in L^∞ , $[M]_*$ denotes the weak star closed linear span of M .

When $H_\varphi^p = ZH^p$ and $|Z| = 1$, put $\mathcal{H}^p =$ the closure (or weak star closure) of polynomials of Z in H^p and put $I^p = \bigcap_{n=1}^\infty Z^n H^p$. Then $H^p = \mathcal{H}^p \oplus I^p$. This is known in [4, Lemma 5].

Throughout this paper, we assume m is the unique representing measure of φ . Then we prove the following theorem.

Theorem . *Let $1 \leq p \leq \infty$ and M is a non-trivial closed subspace in H^p such that $AM \subseteq M$ and $\dim H^p/M = n < \infty$. Then the following are valid.*

- (1) $G(\varphi) = \{\varphi\}$ if and only if $M = H_\varphi^p$.
- (2) $G(\varphi) \neq \{\varphi\}$ if and only if $M = BH^p$ and $B = \prod_{j=1}^n (Z - a_j)/(1 - \overline{a_j}Z)$ where $H_\varphi^p = ZH^p$, $|Z| = 1$ a.e. and $|a_j| < 1$ ($1 \leq j \leq n$).

We need several lemmas in order to prove Theorem.

Lemma 1. *Let M be a closed subspace (or weak star closed) of H^p and $AM \subset M$. If $q > p$ then $\dim H^q/M_q = \dim H^p/M$ and $AM_q \subset M_q$. If $q < p$ then $\dim H^q/[M]_q = \dim H^p/M$ and $A[M]_q \subseteq [M]_q$.*

Proof . It is enough to prove when $q > p$. For if $q < p$ then $[M]_q \cap H^p = M$ by [3, Theorem 6.1]. Suppose $q > p$. Then $AM_q \subseteq M_q$ and M_q is dense in M by [3, Theorem 6.1].

Suppose $\dim H^q/M_q = n < \infty$ and $H^q = [f_j^q : 1 \leq j \leq n]_q \oplus M_q$. If f is nonzero in H^p then there exists $\alpha_{j\ell} \in \mathbb{C}$ and $f_\ell \in M_q$ such that $\|f - (\sum_{j=1}^n \alpha_{j\ell} f_j^q + f_\ell)\|_p \rightarrow 0$ as $\ell \rightarrow \infty$. Hence there exists nonzero $g \in L^{p'}$ ($1/p + 1/p' = 1$) such that g annihilates M_q and $\sum_{j=1}^n \alpha_{j\ell} \int f_j^q g dm \rightarrow \int f g dm$ ($\ell \rightarrow \infty$). Then for any fixed $1 \leq j \leq n$ we can chose g such that $\int f_t^q g dm = 0$ ($t \neq j$) and $\int f_j^q g dm \neq 0$. This implies $\alpha_{j\ell}$ converges α_j . Thus there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ and $f_\infty \in M$ such that $f = \sum_{j=1}^n \alpha_j f_j^q + f_\infty$. Therefore $\dim H^p/M = n$.

Conversely suppose $\dim H^p/M = n < \infty$ and $H^p = [f_j^p : 1 \leq j \leq n]_p \oplus M$. If $\dim H^q/M_q = \ell \leq n$ then by what was proved above, we get $n = \ell$. If $n < \ell \leq \infty$ then $H^q = [f_j^q : 1 \leq j \leq \ell]_q \oplus M_q$. Then $f_j^q = \sum_{t=1}^n \alpha_{tj} f_t^p + f_j$ where $\alpha_{tj} \in \mathbb{C}$ and $f_j \in M$. Hence there exists $\{\beta_j\}_{j=1}^{n+1}$ in $\mathbb{C}$ such that $(\beta_1, \dots, \beta_{n+1}) \neq (0, \dots, 0)$ and $\sum_{j=1}^{n+1} \beta_j (f_j^q - f_j) = \sum_{j=1}^{n+1} (\sum_{t=1}^n \alpha_{tj} f_t^p) \in M$. Hence $\sum_{j=1}^{n+1} \beta_j f_j^q$ belongs to $M \cap H^q = M_q$. This contradicts $H^q = [f_j^q : 1 \leq j \leq \ell]_q \oplus M_q$. $\square$

Lemma 2. *If Theorem is valid for $p = 2$, so it is for all $1 \leq p \leq \infty$.*

Proof . Let M be a closed (or weak star closed) subspace in H^p and $AM \subseteq M$.

Suppose $p < 2$. If $G(\varphi) = \{\varphi\}$, then by hypothesis and Lemma 1 $M_2 = H_\varphi^2$. Since M_2 and H_φ^2 are dense in M and H_φ^p , respectively, $M = H_\varphi^p$. Conversely if $M = H_\varphi^p$ then by Lemma 1 and hypothesis $M_2 = H_\varphi^2$ and so by hypothesis $G(\varphi) = \{\varphi\}$. This shows (1). The statement (2) can be shown similarly.

Suppose $p > 2$. If $G(\varphi) \neq \{\varphi\}$ then by hypothesis and Lemma 1 $[M]_2 = BH^2$ where $B = \prod_{j=1}^n (Z - a_j)/(1 - \bar{a}_j Z)$. By Lemma 1 $M = BH^p$. Conversely if $M = BH^p$ then by Lemma 1 $[M]_2 = BH^2$. By hypothesis this shows $G(\varphi) \neq \{\varphi\}$ and so (2). The statement (1) can be shown similarly. $\square$

Lemma 3. *There exists a unique extention $\tilde{\varphi}$ of φ to H^∞ such that $\tilde{\varphi} \in \mathfrak{M}(H^\infty)$. If $G(\varphi) = \{\varphi\}$ then $G(\tilde{\varphi}) = \{\tilde{\varphi}\}$ and there does not exist a weak star continuous point derivation at $\tilde{\varphi}$.*

Proof . By the definition of m , φ is continuous on A with respect to the weak star topology $\sigma(L^\infty, L^1)$ and so φ has a unique extension $\tilde{\varphi}$ to H^∞ . Then it is easy to see that $\tilde{\varphi} \in \mathfrak{M}(H^\infty)$ and $G(\tilde{\varphi}) = \{\tilde{\varphi}\}$.

If there exists a weak star continuous point derivation at $\tilde{\varphi}$ then $[H_\varphi^\infty H_\varphi^\infty]_* \subsetneq H_\varphi^\infty$. By [3, Chapter V, Theorem 7.2], $H_\varphi^\infty = ZH^\infty$ for some unimodular function Z . Since $G(\tilde{\varphi}) = \{\tilde{\varphi}\}$, $|\tilde{\psi}(Z)| = 1$ for any $\tilde{\psi}$ in $\mathfrak{M}(H^\infty) \setminus \{\tilde{\varphi}\}$. Put $Q = (Z - a)/(1 - \bar{a}Z)$ and $0 < |a| < 1$. Then $\tilde{\varphi}(Q) = -a$ and $|\tilde{\psi}(Q)| = 1$ for any $\tilde{\psi}$ in $\mathfrak{M}(H^\infty) \setminus \{\tilde{\varphi}\}$. This implies Q and $\bar{Q}$ belong to H^∞ and so Q is constant. This contradiction shows there does not exist a weak star continuous point derivation at $\tilde{\varphi}$. $\square$

Lemma 4. *For $j = 1, 2$, let M_j be a closed subspace in H^2 such that $AM_j \subseteq M_j$. If $M_1 \supset M_2$ and $\infty > \dim(M_1 \ominus M_2) \geq 2$ then there exists M_3 such that $M_2 \subsetneq M_3 \subsetneq M_1$ and $AM_3 \subseteq M_3$.*

Proof . For f in A , put $S_f(x) = Q(fx)$ ($x \in M_1 \ominus M_2$) where Q is the orthogonal projection from H^2 to $M_1 \ominus M_2$. Then there exists a function F in A such that $\{0\} \subsetneq \text{Ker} S_F \subsetneq M_1 \ominus M_2$. Since $S_F S_f = S_f S_F$ ($f \in A$), $f(\text{Ker} S_F + M_2) \subseteq \text{Ker} S_F + M_2$ ($f \in A$). Put $M_3 = \text{Ker} S_F + M_2$. $\square$

Proof of Theorem . By Lemma 2 it is enough to prove it only for $p = 2$.

(1) Suppose $G(\varphi) = \{\varphi\}$. By Lemma 1, M_∞ is a weak star closed ideal of H^∞ and $\dim H^\infty/M_\infty = n$. Hence there exists $\tau \in \mathfrak{M}(H^\infty)$ such that $M_\infty \subset \text{Ker} \tau$ and $\text{Ker} \tau$ is weak star closed and $A \text{Ker} \tau \subseteq \text{Ker} \tau$. By Lemma 1, there exists a nonzero function k in H^2 which is orthogonal to $\text{Ker} \tau$. Hence $\tau(f) = \int f k dm$ ($f \in H^\infty$). This implies τ belongs to $G(\tilde{\varphi})$ by [1, Theorem 2.1.1] and so $\tau = \tilde{\varphi}$. Therefore $M \subset H_\varphi^2$ and $\dim H_\varphi^2 \ominus M \leq n - 1$. By Lemma 4, there exists $\tilde{M}$ such that $M \subset \tilde{M} \subset H_\varphi^2$ and $\dim H_\varphi^2 \ominus \tilde{M} = 1$. Hence

by Lemma 1, $\dim H^\infty/\widetilde{M}_\infty = 2$ and $\widetilde{M}_\infty \subset H_\varphi^\infty$. Then $\widetilde{M}_\infty$ is contained in a kernel of a bounded point derivation at $\tilde{\varphi}$ because there does not exist a complex homomorphism τ on H^∞ such that $\widetilde{M}_\infty \subset \text{Ker}\tau$ and $\tau \neq \tilde{\varphi}$ (see [1, Theorem 1.6.1]). This contradicts Lemma 3. Thus $M = H_\varphi^2$.

Conversely suppose $M = H_\varphi^2$. If $G(\varphi) \neq \{\varphi\}$ then $H_\varphi^2 = ZH^2$ by [1, Lemma 4.4.3]. Hence $Z^2H^2 \subseteq H_\varphi^2$, $\dim H_\varphi^2 \ominus Z^2H^2 = 1$ and $AZ^2H^2 \subseteq Z^2H^2$. This contradiction shows $G(\varphi) = \{\varphi\}$.

(2) Suppose $G(\varphi) \neq \{\varphi\}$. Then $H_\varphi^2 = ZH^2$ by [1, Lemma 4.4.3]. If $M = ZM$ then $M \subseteq \bigcap_{\ell=1}^\infty Z^\ell H^2 = I^2$ and so $\dim H^2 \ominus M = \infty$. Hence $M \ominus [ZM]_2 = M \ominus [A_\varphi M]_2 \neq \{0\}$ and so by [3, Chapter V, Theorem 6.2], $M = qH^2$ for some unimodular function q . We shall prove it only when $n \geq 2$ because (2) is clear when $n = 1$. By Lemma 4, there exists a unimodular function q_0 such that $q_0H^2 \supseteq qH^2$ and $\dim (H^2 \ominus q_0H^2) = 1$. By Lemma 1, $\dim H^\infty/q_0H^\infty = 1$. Hence there exists τ in $\mathfrak{M}(H^\infty)$ such that $\text{Ker}\tau$ is weak star closed, $\tau \in G(\tilde{\varphi})$ and $q_0 = (Z - a)/(1 - \bar{a}Z)$ where $\tau(Z) = a$ and $|a| < 1$. If $q_0 \neq q$, again, by Lemma 4, there exists a unimodular function q_1 , such that $q_0H^2 \supseteq q_1H^2 \supseteq qH^2$ and $\dim (q_0H^2 \ominus q_1H^2) = 1$. Then $\overline{q_0}q_1 \in H^\infty$ and so $H^2 \supseteq \overline{q_0}q_1H^2 \supseteq \overline{q_0}qH^2$ and $\dim (H^2 \ominus \overline{q_0}q_1H^2) = 1$. This shows $\overline{q_0}q_1 = (Z - b)/(1 - \bar{b}Z)$ and $|b| < 1$. Hence $q_1 = (Z - a)(Z - b)/(1 - \bar{a}Z)(1 - \bar{b}Z)$. Repeating this process, we get $q = \prod_{j=1}^n (Z - a_j)/(1 - \bar{a}_jZ)$ and $|a_j| < 1$ ($1 \leq j \leq n$).

Now we will prove the converse. Since $H_\varphi^2 = ZH^2$, put $d\mu_a = (1 - |a|^2)/|1 - \bar{a}Z|^2 dm$ where $|a| < 1$. Then $d\mu_a$ is a representing measure of φ_a in $G(\varphi)$ and $\varphi_a \neq \varphi$. $\square$

Corollary 1. *Let M be a non-trivial closed subspace in H^p such that $AM \subseteq M$ and $\dim H^p/M = n < \infty$.*

- (1) *If $G(\varphi) = \{\varphi\}$ then $M = H_\varphi^p$.*
- (2) *If $G(\varphi) \neq \{\varphi\}$ then $M = B\mathcal{H}^p \oplus I^p$ where $B = \prod_{j=1}^n (Z - a_j)/(1 - \bar{a}_jZ)$ and $|a_j| < 1$ ($1 \leq j \leq n$).*

For $1 < p < \infty$, it is known [3, Chapter IV, Theorem 6.2] that a bounded projection P from L^p onto H^p . For a function k in L^∞ , a Hankel operator H_k is defined as follows :

$$H_k f = (I - P)(kf) \quad (f \in H^p).$$

When $p = 2$, if $G(\varphi) = \{\varphi\}$ and H_k is compact then k belongs to H^∞ [2, Theorem 1], and if $G(\varphi) \neq \{\varphi\}$ and H_k is compact then k belongs to the weak * closure of $\bigcup_{n=1}^\infty \overline{Z}^n H^\infty$ [2, Theorem 2]. As a result of Corollary 1, we describe the symbol of finite rank Hankel operator for arbitrary $1 < p < \infty$.

Lemma 5. *If $\text{Ker}H_k$ is the kernel of H_k then $A(\text{Ker}H_k) \subseteq \text{Ker}H_k$.*

Proof . Since $\text{Ker}H_k = \{ f \in H^p : kf \in H^p (k \in L^\infty) \}$, the proof is clear. $\square$

Corollary 2. *Let k be in L^∞ .*

(1) *When $G(\varphi) = \{\varphi\}$, if H_k is of finite rank then $kH^\infty \subseteq H^\infty$ or $kH_\varphi^\infty \subseteq H^\infty$.*

(2) *When $G(\varphi) \neq \{\varphi\}$, if H_k is of finite rank $n (\neq 0)$ then*

$$k = \left(\prod_{j=1}^n \frac{1 - \overline{a_j}Z}{Z - a_j} \right) k_0 + k_1$$

where $|a_j| < 1$ ($1 \leq j \leq n$), k_0 is a nonzero function in $\mathcal{H}^\infty$ and k_1 is a function in I^∞ .

Proof . (1) By Lemma 5, $A(\text{Ker}H_k) \subseteq \text{Ker}H_k$ and so by (1) of Corollary 1, $\text{Ker}H_k = H_\varphi^p$ or H^p . This shows (1).

(2) By Lemma 5 and (2) of Corollary 1, $\text{Ker}H_k = B\mathcal{H}^p \oplus I^p$ where $B = \prod_{j=1}^n (Z - a_j)/(1 - \overline{a_j}Z)$ and $|a_j| < 1$ ($1 \leq j \leq n$). Hence $k(B\mathcal{H}^p \oplus I^p) \subseteq H^p$ and so k belongs to $\overline{B}H^p$. Since $ZI^p = I^p$, it is easy to see $BI^p = I^p$. Therefore k belongs to $\overline{B}H^\infty = \overline{B}\mathcal{H}^\infty \oplus I^\infty$. This shows (2). $\square$

When $p = 2$ and $G(\varphi) = \{\varphi\}$, it is shown [2, Theorem 1] that if H_k is compact and so of finite rank then k belongs to H^∞ . Hence in (1) of Corollary 2, we would like to show that k belongs to H^∞ . Unfortunately we could not prove it.

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