Generalization of Some Inequalities for the Ratio of Gamma Functions

K. Nantomah and M. M. Iddrisu

Department of Mathematics
University for Development Studies
Navrongo Campus, P. O. Box 24
Navrongo, UE/R, Ghana

E. Prempeh

Department of Mathematics
Kwame Nkrumah University of Science and Technology
Kumasi, Ghana

Copyright © 2014 K. Nantomah, M. M. Iddrisu and E. Prempeh. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

We present some monotonic functions and some generalized inequalities involving the ratios of analogues of the Gamma function.

Mathematics Subject Classification: 33B15, 26A48

Keywords: Gamma Function, p-analogue, q-analogue, k-analogue, Inequality

1 Introduction

The classical Euler’s Gamma function $\Gamma(t)$ is commonly defined as

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1} \, dx, \quad t > 0. \quad (1)$$
The $p$-digamma function $\psi_p(t)$, $q$-digamma function $\psi_q(t)$ and $k$-digamma function $\psi_k(t)$ are respectively defined as follows.

$$\psi_p(t) = \frac{d}{dt} \ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad t > 0.$$  \hspace{1cm} (2)

where $\Gamma_p(t)$ is the $p$-analogue of the Gamma function defined by (see [2], [3])

$$\Gamma_p(t) = \frac{p!p^t}{t(t+1)\ldots(t+p)} = \frac{p^t}{t(1+\frac{1}{1})\ldots(1+\frac{1}{p})}, \quad p \in \mathbb{N}, \quad t > 0,$$  \hspace{1cm} (3)

$$\psi_q(t) = \frac{d}{dt} \ln(\Gamma_q(t)) = \frac{\Gamma'_q(t)}{\Gamma_q(t)}, \quad t > 0$$  \hspace{1cm} (4)

where $\Gamma_q(t)$ is the $q$-analogue of the Gamma function defined by (see [4])

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0,$$  \hspace{1cm} (5)

and

$$\psi_k(t) = \frac{d}{dt} \ln(\Gamma_k(t)) = \frac{\Gamma'_k(t)}{\Gamma_k(t)}, \quad t > 0$$  \hspace{1cm} (6)

where $\Gamma_k(t)$ is the $k$-analogue of the Gamma function defined by (see [1], [5])

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{k}{x}}x^{t-1} \, dx, \quad k > 0, \quad t > 0.$$  \hspace{1cm} (7)

In a recent paper [6], Nantomah and Iddrisu proved that the following double inequalities hold:

$$0 < \frac{\Gamma_k(\alpha + t)}{\Gamma_p(\alpha + t)} < \frac{e^{(t-1)(\ln k/\kappa)}}{tp^{1-1}\Gamma_p(\alpha + 1)}, \quad k > 0, \quad p \in \mathbb{N}$$  \hspace{1cm} (8)

and

$$0 < \frac{\Gamma_k(\alpha + t)}{\Gamma_q(\alpha + t)} < \frac{e^{(t-1)(\ln k/\kappa)}}{t(1-q)^{1-t}\Gamma_q(\alpha + 1)}, \quad k > 0, \quad q \in (0,1)$$  \hspace{1cm} (9)

for $t \in (0,1)$ and for a positive real number $\alpha$.

Our objective in this paper is to establish some generalizations of the inequalities (8) and (9).
2 Preliminary Results

The following auxiliary results are crucial to the main results of the paper.

Lemma 2.1. The functions $\psi_p(t)$, $\psi_q(t)$ and $\psi_k(t)$ as defined above have the following series representations.

$$\psi_p(t) = \ln p - \sum_{n=0}^{p} \frac{1}{n+t}, \quad p \in \mathbb{N}, \quad t > 0 \quad (10)$$

$$\psi_q(t) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0 \quad (11)$$

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)}, \quad k > 0, \quad t > 0. \quad (12)$$

where $\gamma$ is the Euler-Mascheroni’s constant.


Lemma 2.2. Let $a > 0$, $b > 0$ and $t > 0$. Then,

$$-a\left(\frac{\ln k - \gamma}{k}\right) + b\ln p + \frac{a}{t} + a\psi_k(t) - b\psi_p(t) > 0.$$ 

Proof. Using the series representations in equations (10) and (12) we have,

$$-a\left(\frac{\ln k - \gamma}{k}\right) + b\ln p + \frac{a}{t} + a\psi_k(t) - b\psi_p(t)$$

$$= a\sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} + b\sum_{n=0}^{p} \frac{1}{(n+t)} > 0.$$

Lemma 2.3. Let $a > 0$, $b > 0$ and $\alpha + \beta t > 0$. Then

$$-a\left(\frac{\ln k - \gamma}{k}\right) + b\ln(1-q) + \frac{a}{t} + a\psi_k(t) - b\psi_q(t) > 0$$

Proof. This follows directly from Lemma 2.2.

Lemma 2.4. Let $a > 0$, $b > 0$ and $t > 0$. Then,

$$-a\left(\frac{\ln k - \gamma}{k}\right) + b\ln(1-q) + \frac{a}{t} + a\psi_k(t) - b\psi_q(t) > 0$$
Proof. Using the series representations in equations (11) and (12) we have,

\[-a\left(\ln\frac{k}{k} - \gamma\right) + b\ln(1 - q) + \frac{a}{t} + a\psi_k(t) - b\psi_q(t)\]

\[= a\sum_{n=1}^{\infty} \frac{t}{nk(nk + t)} - b\ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1 - q^{t+n}} > 0\]

Lemma 2.5. Let \(a > 0\), \(b > 0\) and \(\alpha + \beta t > 0\). Then,

\[-a\left(\ln\frac{k}{k} - \gamma\right) + b\ln(1 - q) + \frac{a}{t} + a\psi_k(\alpha + \beta t) - b\psi_q(\alpha + \beta t) > 0\]

Proof. This follows directly from Lemma 2.4.

3 Main Results

We now state and prove the results of this paper.

Theorem 3.1. Define a function \(\Lambda\) by

\[\Lambda(t) = \frac{t^\alpha e^{-a\beta t} \Gamma_k(\alpha + \beta t)^\alpha}{p^{-b\beta \Gamma_p(\alpha + \beta t)^b}}\]

where \(a, b, \alpha, \beta\) are positive real numbers. Then \(\Lambda\) is increasing on \(t \in (0, \infty)\) and the inequality

\[0 < \frac{\Gamma_k(\alpha + \beta t)^\alpha}{\Gamma_p(\alpha + \beta t)^b} < \frac{e^{a\beta(t-1)\left(\ln\frac{k}{k} - \gamma\right)} \Gamma_k(\alpha + \beta)^a}{t^\alpha p^b(t-1) \Gamma_p(\alpha + \beta)^b}\]

holds for every \(t \in (0, 1)\).

Proof. Let \(g(t) = \ln \Lambda(t)\) for every \(t \in (0, \infty)\). Then,

\[g(t) = \ln \frac{t^\alpha e^{-a\beta t} \Gamma_k(\alpha + \beta t)^\alpha}{p^{-b\beta \Gamma_p(\alpha + \beta t)^b}}\]

\[= -a\beta t\left(\ln\frac{k}{k} - \gamma\right) + b\beta t \ln p + a\beta \ln t + a\ln \Gamma_k(\alpha + \beta t) - b\ln \Gamma_p(\alpha + \beta t)\]

Then,

\[g'(t) = -a\beta\left(\ln\frac{k}{k} - \gamma\right) + b\beta \ln p + \frac{a\beta}{t} + a\beta \psi_k(\alpha + \beta t) - b\beta \psi_q(\alpha + \beta t)\]

\[= \beta \left[-a\left(\ln\frac{k}{k} - \gamma\right) + b\ln p + \frac{a}{t} + a\psi_k(\alpha + \beta t) - b\psi_q(\alpha + \beta t)\right] > 0\]
as a result of Lemma 2.3. This proves that \( g \) is increasing on \( t \in (0, \infty) \). Hence \( \Lambda \) is increasing on \( t \in (0, \infty) \). Thus, for every \( t \in (0, 1) \) we have

\[
\Lambda(0) < \Lambda(t) < \Lambda(1),
\]

yielding

\[
0 < \frac{\Gamma_k(\alpha + \beta t)^a}{\Gamma_p(\alpha + \beta t)^b} < \frac{e^{a\beta(t-1)(\frac{\ln k}{k} - \gamma)}\Gamma_k(\alpha + \beta)^a}{ta\beta p\beta(\alpha - 1)\Gamma_p(\alpha + \beta)^b}.
\]

**Corollary 3.2.** If \( t \in [1, \infty) \), then the following inequality holds.

\[
\frac{e^{a\beta(t-1)(\frac{\ln k}{k} - \gamma)}\Gamma_k(\alpha + \beta)^a}{ta\beta p\beta(\alpha - 1)\Gamma_p(\alpha + \beta)^b} \leq \frac{\Gamma_k(\alpha + \beta t)^a}{\Gamma_p(\alpha + \beta t)^b}.
\]

**Proof.** If \( t \in [1, \infty) \), then we have \( \Lambda(1) \leq \Lambda(t) \) yielding the result.

**Theorem 3.3.** Define a function \( \Upsilon \) by

\[
\Upsilon(t) = \frac{ta\beta e^{-a\beta t(\frac{\ln k}{k} - \gamma)}\Gamma_k(\alpha + \beta t)^a}{(1 - q)b\beta \Gamma_q(\alpha + \beta t)^b}, \quad t \in (0, \infty), \quad k > 0, \quad q \in (0, 1).
\]  

(15)

where \( a, b, \alpha, \beta \) are positive real numbers. Then \( \Upsilon \) is increasing on \( t \in (0, \infty) \) and the inequality

\[
0 < \frac{\Gamma_k(\alpha + \beta t)^a}{\Gamma_q(\alpha + \beta t)^b} < \frac{e^{a\beta(t-1)(\frac{\ln k}{k} - \gamma)}\Gamma_k(\alpha + \beta)^a}{ta\beta(1 - q)^b\beta(\alpha - 1)\Gamma_q(\alpha + \beta)^b}
\]  

(16)

holds for every \( t \in (0, 1) \).

**Proof.** Let \( h(t) = \ln \Upsilon(t) \) for every \( t \in (0, \infty) \). Then,

\[
h(t) = \ln \frac{ta\beta e^{-a\beta t(\frac{\ln k}{k} - \gamma)}\Gamma_k(\alpha + \beta t)^a}{(1 - q)b\beta \Gamma_q(\alpha + \beta t)^b}
\]

\[
= -a\beta t(\frac{\ln k - \gamma}{k}) - b\beta t \ln(1 - q) + a\beta \ln t + a \ln \Gamma_k(\alpha + \beta t) - b \ln \Gamma_q(\alpha + \beta t)
\]

Then,

\[
h'(t) = -a\beta(\frac{\ln k - \gamma}{k}) - b\beta \ln(1 - q) + \frac{a\beta}{t} + a\beta \psi_k(\alpha + \beta t) - b\beta \psi_q(\alpha + \beta t)
\]

\[
= \beta \left[ -a(\frac{\ln k - \gamma}{k}) - b \ln(1 - q) + \frac{a}{t} + a\psi_k(\alpha + \beta t) - b\psi_q(\alpha + \beta t) \right] > 0
\]
as a result of Lemma 2.5. This proves that $h$ is increasing on $t \in (0, \infty)$. Hence $\Upsilon$ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have

$$\Upsilon(0) < \Upsilon(t) < \Upsilon(1)$$

yielding

$$0 < \frac{\Gamma_k(\alpha + \beta t)^a}{\Gamma_q(\alpha + \beta t)^b} < \frac{e^{a\beta(t-1)(\ln k - \gamma_k)}\Gamma_k(\alpha + \beta)^a}{ta\beta(1 - q)^{b\beta(1-t)}\Gamma_q(\alpha + \beta)^b}.$$ 

**Corollary 3.4.** If $t \in [1, \infty)$, then the following inequality holds.

$$\frac{e^{a\beta(t-1)(\ln k - \gamma_k)}\Gamma_k(\alpha + \beta)^a}{ta\beta(1 - q)^{b\beta(1-t)}\Gamma_q(\alpha + \beta)^b} \leq \frac{\Gamma_k(\alpha + \beta t)^a}{\Gamma_q(\alpha + \beta t)^b}.$$ 

**Proof.** If $t \in [1, \infty)$, then we have $\Upsilon(1) \leq \Upsilon(t)$ yielding the result.

## 4 Concluding Remarks

**Remark 4.1.** By putting $a = b = \beta = 1$ into inequalities (14) and (16), we thus obtain respectively, inequalities (8) and (9) as in [6].

## References


*Received: March 19, 2014*