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Fekete Szegö Coefficient for the Janowski α-Spirallike Functions in Open Unit Disk

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Abstract

The aim of this paper is to gives sharp bound of the Fekete Szegö coefficient functional for the Janowski α-Spirallike functions associated with the k^{th} root transformation.

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1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Let $g \in \mathcal{A}$ satisfying the condition

$$e^{i\alpha}\frac{zg'(z)}{g(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)},$$

where $|\alpha| < \frac{\pi}{2}$, $-1 \leq B < A \leq 1$ and $\phi(z)$ is analytic in $\mathbb{U}$ satisfies the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{U}$. Then $g(z)$ is called Janowski $\alpha-$Spirallike functions in the unit disk [7] and we denote such class of functions by $S_\alpha(A, B)$.

Let Ω be the family of function $\phi(z)$ which are regular in the open unit disk $\mathbb{U}$ satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{U}$.

Denote $\mathcal{P}(A, B)$, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ regular in $\mathbb{U}$, such that $p(z) \in \mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 - B\phi(z)}, -1 \leq B < A \leq 1 \tag{1.2}$$

for function $\phi(z) \in \Omega$, and for all $z \in \mathbb{U}$. At the same time, this class can be represented by $\Re(p(z)) > \frac{1-A}{1-B} > 0$.

For real α ($|\alpha| < \frac{\pi}{2}$) a function $f \in \mathcal{A}$ given by (1.1)is said to be in the class of

(1) Janowski α-Spirallike functions $S^*_\alpha(A, B)$ if and only if

$$e^{i\alpha}\frac{zf'(z)}{f(z)} = p(z)\cos\alpha + i\sin\alpha, z \in \mathbb{U} \tag{1.3}$$

(2) $C_\alpha(A, B)$ if and only if

$$e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z)\cos\alpha + i\sin\alpha, \tag{1.4}$$

where $p(z) \in \mathcal{P}(A, B)$.

A function $f \in \mathcal{A}$ is said to be subordinate to $g \in \mathcal{A}$ if there exists a Schwarz function w, analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}),$$

denoted by $f \prec g$ or $f(z) \prec g(z)$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For a univalent function f defined in (1.1), the k^{th} root transform is given by

$$H(z) = [f(z^k)]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.5}$$

Motivated by the earlier works of Ali et al., [2],Koegh et al.,[4] and Polotoglu[7] (see also [10, 11]), in this paper we obtain the sharp bounds for the Fekete-Szegö coefficient functional $\left|b_{2k+1} - \mu b_{k+1}^2\right|$ associated with k^{th} root transform of the function $f(z)$ belongs to the above mentioned classes are derived.

In order to prove our main result we recall the following lemmas regarding the coefficients of functions in Ω. Actually Lemma 1.1 is a reformulation of the results corresponding to the functions with positive real part due to Ma and Minda [6].

Lemma 1.1. *[2] If $w \in \Omega$ and*

$$w(z) = w_1 z + w_2 z^2 + ..., \quad (z \in \mathbb{U}) \tag{1.6}$$

then

$$\left|w_2 - t w_1^2\right| \leq \begin{cases} -t & if\, t \leq -1 \\ 1 & if\, -1 \leq t \leq 1 \\ t & if\, t \geq 1. \end{cases} \tag{1.7}$$

For $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z\frac{\lambda+z}{1+\lambda z}$, $(0 \leq \lambda \leq 1)$ or one of its rotations. While $t = 1$,equality holds if and only if $w(z) = -z\frac{\lambda+z}{1+\lambda z}$,$(0 \leq \lambda \leq 1)$ or one of its rotations.

Lemma 1.2. *[4] If $w \in \Omega$ then $|w_2 - t w_1^2| \leq max\{1, |t|\}$, for any complex number t.*
The result is sharp for the function $w(z) = z$ or $w(z) = z^2$.

Theorem 1.3. *[7, 8]*

$$f(z) \in S_\alpha^*(A, B) \Leftrightarrow \frac{zf'(z)}{f(z)} - 1 \prec \begin{cases} \frac{e^{-i\alpha}(A-B)cos\alpha . z}{1+Bz} & if\, B \neq 0, \\ e^{-i\alpha}(Acos\alpha)z & if\, B = 0. \end{cases} \tag{1.8}$$

2 Coefficient bounds for the k^{th}root transformation

Theorem 2.1. *If f given by (1.1) and belongs to $S^*_\alpha(A,B)$,and $H$is the k^th root transformation of f given by (1.5), then*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \begin{cases} \frac{-B(A-B)}{2k}+\frac{(A-B)^2(1-2\mu)}{2k^2} & \textit{if } \mu\leq\sigma_1, B\neq 0, \\ \frac{A-B}{2k} & \textit{if } \sigma_1\leq\mu\leq\sigma_2, B\neq 0 \\ -\left[\frac{-B(A-B)}{2k}+\frac{(A-B)^2(1-2\mu)}{2k^2}\right] & \textit{if } \mu\geq\sigma_2, B\neq 0 \end{cases}$$

where $\sigma_1=\frac{1}{2}\left[1-\frac{k(B+1)e^{i\alpha}}{(A-B)cos\alpha}\right]$ *and* $\sigma_2=\frac{1}{2}\left[1-\frac{k(B-1)e^{i\alpha}}{(A-B)cos\alpha}\right]$.

Proof. If $f\in S^*_\alpha(A,B)$, then there is an analytic function $w\in\Omega$ of the form (1.6) such that

$$\frac{zf'(z)}{f(z)}-1=\begin{cases}\frac{e^{-i\alpha}(A-B)cos\alpha w(z)}{1+Bw(z)} & \text{if } B\neq 0, \\ e^{-i\alpha}Acos\alpha w(z) & \text{if } B=0.\end{cases}$$

From(1.1),we have

$$\frac{zf'(z)}{f(z)}-1=a_2z+(2a_3-a_2^2)z^2+\dots \tag{2.1}$$

Further,

$$\frac{e^{-i\alpha}(A-B)cos\alpha\ w(z)}{1+Bw(z)}=\frac{e^{-i\alpha}(A-B)cos\alpha\ \left[w_1z+w_2z^2+w_3z^3+\dots\right]}{1+B(w_1z+w_2z^2+w_3z^3+\dots)}$$

$$\frac{e^{-i\alpha}(A-B)cos\alpha\ w(z)}{1+Bw(z)}$$

$$=e^{-i\alpha}(A-B)cos\alpha\ \left[w_1z+(w_2-Bw_1^2)z^2+\dots\right]. \tag{2.2}$$

Equating (2.1)and(2.2), we get

$$a_2=e^{-i\alpha}(A-B)cos\alpha.w_1 \tag{2.3}$$

$$a_3=\frac{1}{2}\left[e^{-i\alpha}(A-B)\cos\alpha\ (w_2-Bw_1^2)+e^{-i2\alpha}(A-B)^2\cos^2\alpha.w_1^2\right]. \tag{2.4}$$

For function f given by (1.1), simple computation yields

$$\left[f(z^k)\right]^{\frac{1}{k}}=z+\frac{1}{k}a_2z^{k+1}+\left(\frac{1}{k}a_3-\frac{1}{2}\frac{k-1}{k^2}a_2^2\right)z^{2k+1}+\dots. \tag{2.5}$$

The equations (1.5) and(2.5) yields

$$b_{k+1} = \frac{1}{k}a_2 \tag{2.6}$$

$$b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}(\frac{k-1}{k^2})a_2^2. \tag{2.7}$$

By using (2.3) and (2.4) in (2.6) and (2.7), it follows

$$b_{k+1} = \frac{1}{k}e^{-i\alpha}(A-B)cos\alpha\ w_1$$

$$b_{2k+1} = \frac{1}{2k}e^{-i\alpha}(A-B)cos\alpha\ [w_2 - Bw_1^2 + e^{-i\alpha}(A-B)cos\alpha\ w_1^2] - \frac{k-1}{k^2}e^{-i2\alpha}(A-B)^2cos^2\alpha\ w_1^2$$

$$= \frac{1}{2k}e^{-i\alpha}(A-B)cos\alpha\ \left[w_2 - Bw_1^2 + \frac{1}{k}e^{-i\alpha}(A-B)cos\alpha\ w_1^2\right]$$

and hence

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{1}{2k}e^{-i\alpha}(A-B)cos\alpha\ \{w_2 - \left[B - e^{-i\alpha}(A-B)cos\alpha\ \frac{(1-2\mu)}{k}\right] w_1^2\}.$$

The first result is established by an application of lemma 1.1,If

$$B - e^{-i\alpha}(A-B)cos\alpha\ \frac{(1-2\mu)}{k} \leq -1$$

then,

$$\mu \leq -\frac{1}{2}\left[\frac{k(B+1)e^{i\alpha}}{(A-B)cos\alpha} - 1\right], (\mu \leq \sigma_1).$$

Lemma 1.1 gives

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \begin{cases} \frac{-B(A-B)}{2k} + \frac{(A-B)^2}{2k^2}(1-2\mu) & \text{if } B \neq 0, \\ \frac{A^2}{2k^2}(1-2\mu) & \text{if } B = 0. \end{cases}$$

If

$$-1 \leq B - e^{-i\alpha}(A-B)cos\alpha.\frac{(1-2\mu)}{k} \leq 1$$

then,

$$\frac{1}{2}\left[1 - \frac{k(B+1)e^{i\alpha}}{(A-B)cos\alpha}\right] \leq \mu \leq \frac{1}{2}\left[1 - \frac{k(B-1)e^{i\alpha}}{(A-B)cos\alpha}\right], (\sigma_1 \leq \mu \leq \sigma_2)$$

and Lemma 1.1 yields

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \begin{cases} \frac{(A-B)}{2k} & \text{if } B \neq 0, \\ \frac{A}{2k} & \text{if } B = 0. \end{cases}$$

If

$$B - e^{-i\alpha}(A-B)cos\alpha \ \frac{(1-2\mu)}{k} \geq 1$$

then,

$$\mu \geq \frac{1}{2}\left[1 - \frac{k(B-1)e^{i\alpha}}{(A-B)cos\alpha}\right], (\mu \geq \sigma_2)$$

and it follows from Lemma 1.1 that

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \begin{cases} -[\frac{-B(A-B)}{2k} + \frac{(A-B)^2}{2k^2}(1-2\mu)] & \text{if } B \neq 0, \\ \frac{-A^2}{2k^2}(1-2\mu) & \text{if } B = 0. \end{cases}$$

The second result follows by an application of Lemma 1.2

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| = \left|\frac{1}{2k}e^{-i\alpha}(A-B)cos\alpha\{w_2 - \left[B - e^{-i\alpha}(A-B)cos\alpha \ \frac{(1-2\mu)}{k}\right] w_1^2\}\right|$$

$$\leq \begin{cases} \frac{(A-B)}{2k}max\{1, \left|B - e^{-i\alpha}(A-B)cos\alpha.\frac{(1-2\mu)}{k}\right|\} & \text{if } B \neq 0, \\ \frac{A}{2k}max\{1, \left|e^{-i\alpha}Acos\alpha.\frac{(1-2\mu)}{k}\right|\} & \text{if } B = 0. \end{cases}$$

□

Taking $k = 1, A = 1, B = -1$ and $\alpha = 0$ we state following :

Corollary 2.2. *If $f \in \mathcal{A}$ satisfies $\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1, \\ -(3-4\mu) & \text{if } \mu \geq 1 \end{cases}$$

Taking $k = 1, A = 1 - 2\beta, B = -1$ and $\alpha = 0$we state following :

Corollary 2.3. *If $f \in \mathcal{A}$ satisfies $\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \beta + 2\beta^2(1-2\mu) & \text{if } \mu \leq \frac{1}{2}, \\ 1 - \beta & \text{if } \frac{1}{2} \leq \mu \leq \frac{1}{1-\beta}, \\ -(\beta + 2\beta^2(1-2\mu)) & \text{if } \mu \geq \frac{1}{1-\beta}. \end{cases}$$

Taking $k = 1, A = 1, B = 0$ and $\alpha = 0$ we state following :

Corollary 2.4. *If $f \in \mathcal{A}$ satisfies $\frac{zf'(z)}{f(z)} \prec 1 + z$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1-2\mu}{2} & \text{if } \mu \leq 0, \\ \frac{1}{2} & \text{if } 0 \leq \mu \leq 1, \\ -(\frac{1-2\mu}{2}) & \text{if } \mu \geq 1. \end{cases}$$

Taking $k = 1, A = \beta, B = 0$ and $\alpha = 0, 0 \leq \beta < 1$ we state following :

Corollary 2.5. *If $f \in \mathcal{A}$ satisfies $\frac{zf'(z)}{f(z)} \prec 1 + \beta z$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \beta^2(\frac{1-2\mu}{2}) & \text{if } \mu \leq \frac{\beta-1}{2\beta}, \\ 1 - \beta & \text{if } \frac{\beta-1}{2\beta} \leq \mu \leq \frac{\beta+1}{2\beta}, \\ -(\beta^2(\frac{1-2\mu}{2})) & \text{if } \mu \geq \frac{\beta+1}{2\beta}. \end{cases}$$

Taking $k = 1, A = \beta, B = -\beta$ and $\alpha = 0, 0 \leq \beta < 1$ we state following :

Corollary 2.6. *If $f \in \mathcal{A}$ satisfies $\frac{zf'(z)}{f(z)} \prec \frac{1+\beta z}{1-\beta z}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \beta^2(3 - 4\mu) & \text{if } \mu \leq \frac{3\beta-1}{4\beta}, \\ \beta & \text{if } \frac{3\beta-1}{4\beta} \leq \mu \leq \frac{3\beta+1}{4\beta}, \\ -(\beta^2(3 - 4\mu)) & \text{if } \mu \geq \frac{3\beta+1}{4\beta}. \end{cases}$$

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