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On W_4-Flatness of Some Classes of Generalizations of Einstein Manifolds

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Abstract

In this paper, we studied generalizations of quasi Einstein spaces admitting the W_4-curvature tensor. We obtained necessary conditions for W_4-flatness of quasi Einstein, $N(k)$-quasi Einstein, generalized quasi Einstein, pseudo generalized quasi Einstein, $N(k)$-mixed quasi Einstein, mixed generalized quasi Einstein, super quasi Einstein and mixed super quasi Einstein manifolds.

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1 Preliminaries

M.C. Chaki and R.K .Maity [2] introduced quasi Einstein manifolds. A non-flat Riemannian manifold (M^n, g), $n > 2$, is called quasi Einstein if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1}$$

where α and β are scalar functions of which $\beta \neq 0$ and A is a nonzero 1-form such that $g(X, \rho) = A(X)$, for all vector fields X with ρ being a unit vector field called generator of the manifold. An n-dimensional manifold of this kind is denoted by $(QE)_n$. M.M. Tripathy and J.S. Kim [11] introduced the notion of $N(k)$-quasi Einstein manifold which is defined as a quasi Einstein manifold whose generator ρ belongs to the k-nullity distribution $N(k)$ of the manifold. The k-nullity distribution $N(k)$ of a Riemannian manifold M [10] is given by

$$N(k) : p \to N_p(k) = \{Z \in T_pM : R(X, Y)Z = k\,(g(Y, Z)X - g(X, Z)Y)\}\,,$$

for all X, $Y \in TM$, where k is some smooth function and R is the curvature tensor.

Motivated by this definition, H.G. Nagaraja [8] defined $N(k)$-mixed quasi Einstein manifold as a non flat Riemannian manifold (M^n, g), $n > 2$, whose Ricci tensor S is nonzero and satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)B(Y) + \gamma A(Y)B(X), \tag{2}$$

where α, β, γ are scalar functions of which $\beta \neq 0$ and $\gamma \neq 0$, A and B are non-zero 1-forms such that $g(X, \rho) = A(X)$, $g(X, \mu) = B(X)$ for all vector fields X, with ρ and μ being orthogonal unit vector fields called generators of the manifold belong to the k-nullity distribution $N(k)$.

There have been several recent studies on generalizations of quasi Einstein spaces on which we will focus in this paper. In particular, we consider the following:

(i) A non-flatRiemannian manifold (M^n, g), $n > 2$, is called generalized quasi Einstein [3] if its Ricci tensor S of type (0,2) is nonzero and satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \tag{3}$$

where α, β, γ are scalar functions of which $\beta \neq 0$ and $\gamma \neq 0$, A and B are non-zero 1-forms with $g(X, \rho) = A(X)$, $g(X, \mu) = B(X)$ for all vector fields X, with ρ and μ being two orthogonal unit vector fields called generators of the manifold.

(ii) A non-flat Riemannian manifold (M^n, g), $n > 2$, is called pseudo generalized quasi Einstein if its Ricci tensor S of type (0,2) is nonzero and satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta D(X, Y), \tag{4}$$

where α, β, γ, δ are scalar functions of which $\beta \neq 0$, $\gamma \neq 0$ and $\delta \neq 0$, A and B are non-zero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields X, with ρ and μ being orthogonal unit vector fields, and D is a symmetric 2-form with zero trace such that $D(X,\rho) = 0$ for all X.

(iii) A non-flat Riemannian manifold (M^n, g), $n > 2$, is called mixed generalized quasi Einstein if its Ricci tensor S of type (0,2) is nonzero and satisfies

$$\begin{aligned} S(X,Y) &= \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) \\ &\quad + \delta\left[A(X)B(Y) + A(Y)B(X)\right], \end{aligned} \tag{5}$$

where α, β, γ, δ are scalar functions of which $\beta \neq 0$, $\gamma \neq 0$ and $\delta \neq 0$, A and B are non-zero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields X, with ρ and μ being orthogonal unit vector fields.

(iv) A non-flat Riemannian manifold (M^n, g), $n > 2$, is called super quasi Einstein if its Ricci tensor S of type (0,2) is nonzero and satisfies

$$\begin{aligned} S(X,Y) &= \alpha g(X,Y) + \beta A(X)A(Y) \\ &\quad + \gamma\left[A(X)B(Y) + A(Y)B(X)\right] + \delta D(X,Y), \end{aligned} \tag{6}$$

where α, β, γ, δ are scalar functions of which $\beta \neq 0$, $\gamma \neq 0$ and $\delta \neq 0$, A and B are non-zero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields X, with ρ and μ being two orthogonal unit vector fields, and D is a symmetric 2-form with zero trace such that $D(X,\rho) = 0$ for all vector fields X.

(v) A non-flat Riemannian manifold (M^n, g), $n > 2$, is called mixed super quasi Einstein if its Ricci tensor S of type (0,2) is nonzero and satisfies

$$\begin{aligned} S(X,Y) &= \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) \\ &\quad + \delta\left[A(X)B(Y) + A(Y)B(X)\right] + \epsilon D(X,Y), \end{aligned} \tag{7}$$

where α, β, γ, δ, ϵ are scalar functions of which $\beta \neq 0$, $\gamma \neq 0$, $\delta \neq 0$ and $\epsilon \neq 0$, A and B are non-zero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields X, with ρ and μ being two orthogonal unit vector fields, and D is a symmetric 2-form with zero trace such that $D(X,\rho) = 0$ for all vector fields X.

In 1972, Pokhariyal defined several curvature tensors [9] and studied their physical and geometric properties. One of the tensors he introduced was the

W_4-curvature tensor. As a $(0,4)$-tensor, it is given by

$$W_4(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\{g(X,Z)S(Y,T) - g(X,Y)S(Z,T)\}, \tag{8}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor. This has no symmetry but it satisfies the cyclic property

$$W_4(X,Y,Z,T) + W_4(Y,X,Z,T) + W_4(Z,X,Y,T) = 0,$$

and on contraction, it reduces to the Ricci tensor, *i.e.*, $W_{4ij} = R_{ij}$. Pokhariyal also showed that the vanishing of the divergence of $W_{4\,ijk}^{\;h}$ in an electromagnetic field implies a purely electric field.

From (8), we can see that if a manifold is W_4-flat, *i.e.*, $W_4(X,Y,Z,T) = 0$ for all X, Y, Z, $T \in TM$, then the Riemannian curvature satisfies

$$R(X,Y,Z,T) = \frac{1}{n-1}\{g(X,Y)S(Z,T) - g(X,Z)S(Y,T)\}. \tag{9}$$

2 Results

Theorem 2.1. *A W_4-flat quasi Einstein or W_4-flat $N(k)$-quasi Einstein manifold is an Einstein manifold.*

Proof. Suppose we have a W_4-flat quasi Einstein or $N(k)$-quasi Einstein manifold. Then, from equations (1) and (9), we have

$$\begin{aligned} R(X,Y,Z,\rho) &= \frac{1}{n-1}\{g(X,Y)S(Z,\rho) - g(X,Z)S(Y,\rho)\} \\ &= \frac{1}{n-1}\{(\alpha+\beta)A(Z)g(X,Y) - (\alpha+\beta)A(Y)g(X,Z)\} \end{aligned} \tag{10}$$

and

$$\begin{aligned} R(Z,\rho,X,Y) &= \frac{1}{n-1}\{g(Z,\rho)S(X,Y) - g(Z,X)S(\rho,Y)\} \\ &= \frac{1}{n-1}\{\alpha A(Z)g(X,Y) + \beta A(X)A(Y)A(Z) \\ &\qquad -(\alpha+\beta)A(Y)g(X,Z)\}. \end{aligned} \tag{11}$$

Since $R(X,Y,Z,\rho) = R(Z,\rho,X,Y)$, $\beta \neq 0$ and A is a nonzero 1-form, then equations (10) and (11) give

$$g(X,Y) = A(X)A(Y).$$

Thus, the Ricci tensor reduces to

$$S(X,Y) = cg(X,Y),$$

where $c = \alpha+\beta$, that is, the manifold in consideration is an Einstein manifold. □

Theorem 2.2. *In a W_4-flat generalized quasi Einstein manifold, the scalars β and γ are equal.*

Proof. Consider a W_4-flat generalized quasi Einstein manifold. Using equations (3) and (9), we get

$$\begin{aligned} R(X,\mu,\rho,Y) &= \frac{1}{n-1}\{g(X,\mu)S(\rho,Y) - g(X,\rho)S(\mu,Y)\} \\ &= \frac{1}{n-1}\{(\alpha+\beta)A(Y)B(X) - (\alpha+\gamma)A(X)B(Y)\} \end{aligned} \tag{12}$$

and

$$\begin{aligned} R(\rho,Y,X,\mu) &= \frac{1}{n-1}\{g(\rho,Y)S(X,\mu) - g(\rho,X)S(Y,\mu)\} \\ &= \frac{1}{n-1}\{(\alpha+\gamma)\left[A(Y)B(X) - A(X)B(Y)\right]\}. \end{aligned} \tag{13}$$

Since $R(X,\mu,\rho,Y) = R(\rho,Y,X,\mu)$, then equations (12) and (13) imply

$$(\beta-\gamma)A(Y)B(X) = 0.$$

But A and B are nonzero 1-forms. Hence, $\beta = \gamma$. □

Theorem 2.3. *In a W_4-flat pseudo generalized quasi Einstein manifold, either $A(X)B(Y) = -A(Y)B(X)$ or $\alpha = -\beta$.*

Proof. Consider a W_4-flat pseudo generalized quasi Einstein manifold. Using equations (4) and (9), we obtain the following:

$$\begin{aligned} R(X,Y,\rho,\mu) &= \frac{1}{n-1}\{g(X,Y)S(\rho,\mu)-g(X,\rho)S(Y,\mu)\} \\ &= \frac{1}{n-1}\{-(\alpha+\gamma)A(X)B(Y)-\delta A(X)D(Y,\mu)\} \qquad (14) \\ R(X,Y,\mu,\rho) &= \frac{1}{n-1}\{g(X,Y)S(\mu,\rho)-g(X,\mu)S(Y,\rho)\} \\ &= -\frac{(\alpha+\beta)}{n-1}A(Y)B(X) \qquad (15) \\ R(X,\rho,\mu,Y) &= \frac{1}{n-1}\{g(X,\rho)S(\mu,Y)-g(X,\mu)S(\rho,Y)\} \\ &= \frac{1}{n-1}\{(\alpha+\gamma)A(X)B(Y)-(\alpha+\beta)A(Y)B(X) \\ &\qquad +\delta A(X)D(\mu,Y)\} \qquad (16) \\ R(\mu,Y,X,\rho) &= \frac{1}{n-1}\{g(\mu,Y)S(X,\rho)-g(\mu,X)S(Y,\rho)\} \\ &= \frac{(\alpha+\beta)}{n-1}[A(X)B(Y)-A(Y)B(X)] \qquad (17) \end{aligned}$$

From equations (14) and (15), since $R(X,Y,\rho,\mu)=-R(X,Y,\mu,\rho)$, we have

$$\delta A(X)D(Y,\mu)=-(\alpha+\gamma)A(X)B(Y)-(\alpha+\beta)A(Y)B(X). \qquad (18)$$

Meanwhile, from equations (16) and (17), since $R(X,\rho,\mu,Y)=R(\mu,Y,X,\rho)$, we get

$$\delta A(X)D(\mu,Y)=(\beta-\gamma)A(X)B(Y). \qquad (19)$$

Combining equations (18) and (19), we yield

$$(\alpha+\beta)\left(A(X)B(Y)+A(Y)B(X)\right)=0.$$

Thus, $\alpha=-\beta$ or $A(X)B(Y)=-A(Y)B(X)$. □

Theorem 2.4. *In a W_4-flat $N(k)$-mixed quasi Einstein manifold, $A(X)A(Y)$ is a multiple of $B(X)B(Y)$, i.e., $A(X)A(Y)=cB(X)B(Y)$. Here, $c=\frac{\beta}{\gamma}$.*

Proof. Consider a W_4-flat $N(k)$-mixed quasi Einstein manifold. Then, from (2) and (9), we obtain

$$\begin{aligned} R(X,\rho,\mu,Y) &= \frac{1}{n-1}\{g(X,\rho)S(\mu,Y)-g(X,\mu)S(\rho,Y)\} \\ &= \frac{1}{n-1}\{\alpha[A(X)B(Y)-A(Y)B(X)] \\ &\qquad +\gamma A(X)A(Y)-\beta B(X)B(Y)\} \qquad (20) \\ R(\mu,Y,X,\rho) &= \frac{1}{n-1}\{g(\mu,Y)S(X,\rho)-g(\mu,X)S(Y,\rho)\} \\ &= \frac{\alpha}{n-1}[A(X)B(Y)-A(Y)B(X)]. \qquad (21) \end{aligned}$$

Since $R(X,\rho,\mu,Y) = R(\mu,Y,X,\rho)$, using (20) and (21), we have

$$\gamma A(X)A(Y) = \beta B(X)B(Y).$$

□

Theorem 2.5. *There does not exist W_4-flat mixed generalized quasi Einstein manifold.*

Proof. Suppose we have a W_4-flat mixed generalized quasi Einstein manifold. Using equations (5) and (9), we have

$$\begin{aligned} R(X,Y,\rho,\mu) &= \frac{1}{n-1}\{g(X,Y)S(\rho,\mu) - g(X,\rho)S(Y,\mu)\} \\ &= \frac{1}{n-1}\{\delta g(X,Y) - A(X)[(\alpha+\gamma)B(Y) + \delta A(Y)]\} \quad (22) \\ R(\rho,\mu,X,Y) &= \frac{1}{n-1}\{g(\rho,\mu)S(X,Y) - g(\rho,X)S(\mu,Y)\} \\ &= \frac{1}{n-1}\{-A(X)[(\alpha+\gamma)B(Y) + \delta A(Y)]\}. \quad (23) \end{aligned}$$

Note that $R(X,Y,\rho,\mu) = R(\rho,\mu,X,Y)$. So from equations (22) and (23), we obtain

$$\delta g(X,Y) = 0, \text{ for all } X,Y \in TM.$$

This implies $g(X,Y) = 0$, $\forall X,Y$, since $\delta \neq 0$. This is a contradiction. □

Theorem 2.6. *There does not exist W_4-flat super quasi Einstein manifold.*

Proof. Suppose we have a W_4-flat super quasi Einstein manifold. Using equations (6) and (9), we get

$$\begin{aligned} R(X,Y,\rho,\mu) &= \frac{1}{n-1}\{g(X,Y)S(\rho,\mu) - g(X,\rho)S(Y,\mu)\} \\ &= \frac{1}{n-1}\{\gamma g(X,Y) - A(X)[\alpha B(Y) \\ &\qquad +\gamma A(Y) + \delta D(Y,\mu)]\} \quad (24) \\ R(\rho,\mu,X,Y) &= \frac{1}{n-1}\{g(\rho,\mu)S(X,Y) - g(\rho,X)S(\mu,Y)\} \\ &= \frac{1}{n-1}\{-A(X)[\alpha B(Y) + \gamma A(Y) + \delta D(\mu,Y)]\}. \quad (25) \end{aligned}$$

Since D is symmetric and $R(X,Y,\rho,\mu) = R(\rho,\mu,X,Y)$, equations (24) and (25) imply

$$\gamma g(X,Y) = 0, \text{ for all } X,Y \in TM.$$

This implies $g(X,Y) = 0$, $\forall X,Y$, since $\gamma \neq 0$. This is a contradiction. □

Theorem 2.7. *There does not exist W_4-flat mixed super quasi Einstein manifold.*

Proof. Suppose we have a W_4-flat mixed super quasi Einstein manifold. Using equations (7) and (9), we yield

$$\begin{aligned} R(X,Y,\rho,\mu) &= \frac{1}{n-1}\{g(X,Y)S(\rho,\mu) - g(X,\rho)S(Y,\mu)\} \\ &= \frac{1}{n-1}\{\delta g(X,Y) - A(X)\left[(\alpha+\gamma)B(Y)\right. \\ &\qquad \left.+\delta A(Y) + \epsilon D(Y,\mu)\right]\} \end{aligned} \tag{26}$$

$$\begin{aligned} R(\rho,\mu,X,Y) &= \frac{1}{n-1}\{g(\rho,\mu)S(X,Y) - g(\rho,X)S(\mu,Y)\} \\ &= \frac{1}{n-1}\{-A(X)\left[(\alpha+\gamma)B(Y) + \delta A(Y) + \epsilon D(\mu,Y)\right]\} \end{aligned} \tag{27}$$

Since D is symmetric and $R(X,Y,\rho,\mu) = R(\rho,\mu,X,Y)$, equations (26) and (27) imply

$$\delta g(X,Y) = 0, \text{ for all } X,Y \in TM.$$

But $\delta \neq 0$, so $g(X,Y) = 0$, for all $X,Y \in TM$. This is a contradiction. □

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