On $W_4$-Flatness of Some Classes of Generalizations of Einstein Manifolds

Dennis T. Leyson

Institute of Mathematics and Natural Sciences Research Institute
College of Science, University of the Philippines
Diliman, Quezon City, 1101 Philippines

Richard S. Lemence

Institute of Mathematics and Natural Sciences Research Institute
College of Science, University of the Philippines
Diliman, Quezon City, 1101 Philippines

Copyright © 2014 Dennis T. Leyson and Richard S. Lemence. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we studied generalizations of quasi Einstein spaces admitting the $W_4$-curvature tensor. We obtained necessary conditions for $W_4$-flatness of quasi Einstein, $N(k)$-quasi Einstein, generalized quasi Einstein, pseudo generalized quasi Einstein, mixed generalized quasi Einstein, mixed generalized quasi Einstein, super quasi Einstein and mixed super quasi Einstein manifolds.

Mathematics Subject Classification: 53C25

Keywords: Einstein manifold, quasi Einstein manifold, $W_4$-curvature tensor, $W_4$-flat manifold

1 Preliminaries

M.C. Chaki and R.K. Maity [2] introduced quasi Einstein manifolds. A non-flat Riemannian manifold $(M^n, g)$, $n > 2$, is called quasi Einstein if its Ricci
tensor $S$ of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y),$$  \hspace{1cm} (1)

where $\alpha$ and $\beta$ are scalar functions of which $\beta \neq 0$ and $A$ is a nonzero 1-form such that $g(X,\rho) = A(X)$, for all vector fields $X$ with $\rho$ being a unit vector field called generator of the manifold. An $n$-dimensional manifold of this kind is denoted by $(QE)_n$. M.M. Tripathy and J.S. Kim [11] introduced the notion of $N(k)$-quasi Einstein manifold which is defined as a quasi Einstein manifold whose generator $\rho$ belongs to the $k$-nullity distribution $N(k)$ of the manifold. The $k$-nullity distribution $N(k)$ of a Riemannian manifold $M$ [10] is given by

$$N(k) : p \to N_p(k) = \{Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\},$$

for all $X, Y \in TM$, where $k$ is some smooth function and $R$ is the curvature tensor.

Motivated by this definition, H.G. Nagaraja [8] defined $N(k)$-mixed quasi Einstein manifold as a non flat Riemannian manifold $(M^n, g)$, $n > 2$, whose Ricci tensor $S$ is nonzero and satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)B(Y) + \gamma A(Y)B(X),$$  \hspace{1cm} (2)

where $\alpha, \beta, \gamma$ are scalar functions of which $\beta \neq 0$ and $\gamma \neq 0$, $A$ and $B$ are non-zero 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields $X$, with $\rho$ and $\mu$ being orthogonal unit vector fields called generators of the manifold belong to the $k$-nullity distribution $N(k)$.

There have been several recent studies on generalizations of quasi Einstein spaces on which we will focus in this paper. In particular, we consider the following:

(i) A non-flat Riemannian manifold $(M^n, g)$, $n > 2$, is called generalized quasi Einstein [3] if its Ricci tensor $S$ of type $(0,2)$ is nonzero and satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y),$$  \hspace{1cm} (3)

where $\alpha, \beta, \gamma$ are scalar functions of which $\beta \neq 0$ and $\gamma \neq 0$, $A$ and $B$ are non-zero 1-forms with $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields $X$, with $\rho$ and $\mu$ being two orthogonal unit vector fields called generators of the manifold.

(ii) A non-flat Riemannian manifold $(M^n, g)$, $n > 2$, is called pseudo generalized quasi Einstein if its Ricci tensor $S$ of type $(0,2)$ is nonzero and satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta D(X,Y),$$  \hspace{1cm} (4)
where $\alpha, \beta, \gamma, \delta$ are scalar functions of which $\beta \neq 0, \gamma \neq 0$ and $\delta \neq 0$, $A$ and $B$ are non-zero 1-forms such that $g(X, \rho) = A(X), g(X, \mu) = B(X)$ for all vector fields $X$, with $\rho$ and $\mu$ being orthogonal unit vector fields, and $D$ is a symmetric 2-form with zero trace such that $D(X, \rho) = 0$ for all $X$.

(iii) A non-flat Riemannian manifold $(M^n, g)$, $n > 2$, is called mixed generalized quasi Einstein if its Ricci tensor $S$ of type $(0,2)$ is nonzero and satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta [A(X)B(Y) + A(Y)B(X)],$$

where $\alpha, \beta, \gamma, \delta$ are scalar functions of which $\beta \neq 0, \gamma \neq 0$ and $\delta \neq 0$, $A$ and $B$ are non-zero 1-forms such that $g(X, \rho) = A(X), g(X, \mu) = B(X)$ for all vector fields $X$, with $\rho$ and $\mu$ being orthogonal unit vector fields.

(iv) A non-flat Riemannian manifold $(M^n, g)$, $n > 2$, is called super quasi Einstein if its Ricci tensor $S$ of type $(0,2)$ is nonzero and satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta [A(X)B(Y) + A(Y)B(X)] + \epsilon D(X, Y),$$

where $\alpha, \beta, \gamma, \delta$ are scalar functions of which $\beta \neq 0, \gamma \neq 0$ and $\delta \neq 0$, $A$ and $B$ are non-zero 1-forms such that $g(X, \rho) = A(X), g(X, \mu) = B(X)$ for all vector fields $X$, with $\rho$ and $\mu$ being two orthogonal unit vector fields, and $D$ is a symmetric 2-form with zero trace such that $D(X, \rho) = 0$ for all vector fields $X$.

(v) A non-flat Riemannian manifold $(M^n, g)$, $n > 2$, is called mixed super quasi Einstein if its Ricci tensor $S$ of type $(0,2)$ is nonzero and satisfies

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta [A(X)B(Y) + A(Y)B(X)] + \epsilon D(X, Y),$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are scalar functions of which $\beta \neq 0, \gamma \neq 0, \delta \neq 0$ and $\epsilon \neq 0$, $A$ and $B$ are non-zero 1-forms such that $g(X, \rho) = A(X), g(X, \mu) = B(X)$ for all vector fields $X$, with $\rho$ and $\mu$ being two orthogonal unit vector fields, and $D$ is a symmetric 2-form with zero trace such that $D(X, \rho) = 0$ for all vector fields $X$.

In 1972, Pokhariyal defined several curvature tensors [9] and studied their physical and geometric properties. One of the tensors he introduced was the
$W_4$-curvature tensor. As a $(0,4)$-tensor, it is given by

$$W_4(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1} \{g(X,Z)S(Y,T) - g(X,Y)S(Z,T)\},$$

(8)

where $R$ is the Riemannian curvature tensor and $S$ is the Ricci tensor. This has no symmetry but it satisfies the cyclic property


and on contraction, it reduces to the Ricci tensor, i.e., $W_{4ij} = R_{ij}$. Pokhariyal also showed that the vanishing of the divergence of $W_{4ijk}^h$ in an electromagnetic field implies a purely electric field.

From (8), we can see that if a manifold is $W_4$-flat, i.e., $W_4(X,Y,Z,T) = 0$ for all $X, Y, Z, T \in TM$, then the Riemannian curvature satisfies

$$R(X,Y,Z,T) = \frac{1}{n-1} \{g(X,Y)S(Z,T) - g(X,Z)S(Y,T)\}. \quad (9)$$

## 2 Results

**Theorem 2.1.** A $W_4$-flat quasi Einstein or $W_4$-flat $N(k)$-quasi Einstein manifold is an Einstein manifold.

**Proof.** Suppose we have a $W_4$-flat quasi Einstein or $N(k)$-quasi Einstein manifold. Then, from equations (1) and (9), we have

$$R(X,Y,Z,\rho) = \frac{1}{n-1} \{g(X,Y)S(Z,\rho) - g(X,Z)S(Y,\rho)\}$$

$$= \frac{1}{n-1} \{\frac{1}{\alpha + \beta}A(Z)g(X,Y) - (\alpha + \beta)A(Y)g(X,Z)\} \quad (10)$$

and

$$R(Z,\rho,X,Y) = \frac{1}{n-1} \{g(Z,\rho)S(X,Y) - g(Z,X)S(\rho,Y)\}$$

$$= \frac{1}{n-1} \{\alpha A(Z)g(X,Y) + \beta A(X)A(Y)A(Z) - (\alpha + \beta)A(Y)g(X,Z)\}. \quad (11)$$

Since $R(X,Y,Z,\rho) = R(Z,\rho,X,Y)$, $\beta \neq 0$ and $A$ is a nonzero 1-form, then equations (10) and (11) give

$$g(X,Y) = A(X)A(Y).$$
Thus, the Ricci tensor reduces to

\[ S(X, Y) = cg(X, Y), \]

where \( c = \alpha + \beta \), that is, the manifold in consideration is an Einstein manifold.

**Theorem 2.2.** In a \( W_4 \)-flat generalized quasi Einstein manifold, the scalars \( \beta \) and \( \gamma \) are equal.

**Proof.** Consider a \( W_4 \)-flat generalized quasi Einstein manifold. Using equations (3) and (9), we get

\[
R(X, \mu, \rho, Y) = \frac{1}{n-1} \{ g(X, \mu)S(\rho, Y) - g(X, \rho)S(\mu, Y) \} = \frac{1}{n-1} \{ (\alpha + \beta)A(Y)B(X) - (\alpha + \gamma)A(X)B(Y) \} \quad (12)
\]

and

\[
R(\rho, Y, X, \mu) = \frac{1}{n-1} \{ g(\rho, Y)S(X, \mu) - g(\rho, X)S(Y, \mu) \} = \frac{1}{n-1} \{ (\alpha + \gamma) [A(Y)B(X) - A(X)B(Y)] \}. \quad (13)
\]

Since \( R(X, \mu, \rho, Y) = R(\rho, Y, X, \mu) \), then equations (12) and (13) imply

\[
(\beta - \gamma)A(Y)B(X) = 0.
\]

But \( A \) and \( B \) are nonzero 1-forms. Hence, \( \beta = \gamma \). 

**Theorem 2.3.** In a \( W_4 \)-flat pseudo generalized quasi Einstein manifold, either \( A(X)B(Y) = -A(Y)B(X) \) or \( \alpha = -\beta \).

**Proof.** Consider a \( W_4 \)-flat pseudo generalized quasi Einstein manifold. Using
Proof. Consider a

In a

Theorem 2.4. Combining equations (18) and (19), we yield

equations (4) and (9), we obtain the following:

$$R(X, Y, \rho, \mu) = \frac{1}{n-1} \{g(X, Y)S(\rho, \mu) - g(X, \rho)S(Y, \mu)\}$$

$$= \frac{1}{n-1} \{-(\alpha + \gamma)A(X)B(Y) - \delta A(X)D(Y, \mu)\} \quad (14)$$

$$R(X, Y, \mu, \rho) = \frac{1}{n-1} \{g(X, Y)S(\mu, \rho) - g(X, \mu)S(Y, \rho)\}$$

$$= -\frac{(\alpha + \beta)}{n-1}A(Y)B(X) \quad (15)$$

$$R(X, \rho, \mu, Y) = \frac{1}{n-1} \{g(X, \rho)S(\mu, Y) - g(X, \mu)S(\rho, Y)\}$$

$$= \frac{1}{n-1} \{((\alpha + \gamma)A(X)B(Y) - (\alpha + \beta)A(Y)B(X)$$

$$+ \delta A(X)D(\mu, Y)\} \quad (16)$$

$$R(\mu, Y, X, \rho) = \frac{1}{n-1} \{g(\mu, Y)S(X, \rho) - g(\mu, X)S(Y, \rho)\}$$

$$= -\frac{(\alpha + \beta)}{n-1} [A(X)B(Y) - A(Y)B(X)] \quad (17)$$

From equations (14) and (15), since

$$\delta A(X)D(\mu, Y) = -(\alpha + \gamma)A(X)B(Y) - (\alpha + \beta)A(Y)B(X). \quad (18)$$

Meanwhile, from equations (16) and (17), since

$$R(X, \rho, \mu, Y) = R(\mu, Y, X, \rho),$$

we get

$$\delta A(X)D(\mu, Y) = (\beta - \gamma)A(X)B(Y). \quad (19)$$

Combining equations (18) and (19), we yield

$$(\alpha + \beta) (A(X)B(Y) + A(Y)B(X)) = 0.$$ 

Thus, \(\alpha = -\beta\) or \(A(X)B(Y) = -A(Y)B(X). \quad \square$$

**Theorem 2.4.** In a \(W_4\)-flat \(N(k)\)-mixed quasi Einstein manifold, \(A(X)A(Y)\) is a multiple of \(B(X)B(Y)\), i.e., \(A(X)A(Y) = cB(X)B(Y)\). Here, \(c = \frac{2}{\gamma} \).

**Proof.** Consider a \(W_4\)-flat \(N(k)\)-mixed quasi Einstein manifold. Then, from

(2) and (9), we obtain

$$R(X, \rho, \mu, Y) = \frac{1}{n-1} \{g(X, \rho)S(\mu, Y) - g(X, \mu)S(\rho, Y)\}$$

$$= \frac{1}{n-1} \{\alpha [A(X)B(Y) - A(Y)B(X)]$$

$$+ \gamma A(X)A(Y) - \beta B(X)B(Y)\} \quad (20)$$

$$R(\mu, Y, X, \rho) = \frac{1}{n-1} \{g(\mu, Y)S(X, \rho) - g(\mu, X)S(Y, \rho)\}$$

$$= \frac{\alpha}{n-1} [A(X)B(Y) - A(Y)B(X)]. \quad (21)$$
Since $R(X, \rho, \mu, Y) = R(\mu, Y, X, \rho)$, using (20) and (21), we have
\[ \gamma A(X)A(Y) = \beta B(X)B(Y). \]

\[ \square \]

**Theorem 2.5.** There does not exist $W_4$-flat mixed generalized quasi Einstein manifold.

**Proof.** Suppose we have a $W_4$-flat mixed generalized quasi Einstein manifold. Using equations (5) and (9), we have
\[ R(X, Y, \rho, \mu) = \frac{1}{n-1} \{ g(X, Y)S(\rho, \mu) - g(X, \rho)S(Y, \mu) \} \]
\[ = \frac{1}{n-1} \{ \delta g(X, Y) - A(X) [(\alpha + \gamma)B(Y) + \delta A(Y)] \} \quad (22) \]
\[ R(\rho, \mu, X, Y) = \frac{1}{n-1} \{ g(\rho, \mu)S(X, Y) - g(\rho, X)S(\mu, Y) \} \]
\[ = \frac{1}{n-1} \{ -A(X) [(\alpha + \gamma)B(Y) + \delta A(Y)] \} . \quad (23) \]

Note that $R(X, Y, \rho, \mu) = R(\rho, \mu, X, Y)$. So from equations (22) and (23), we obtain
\[ \delta g(X, Y) = 0, \text{ for all } X, Y \in TM. \]
This implies $g(X, Y) = 0, \forall X, Y$, since $\delta \neq 0$. This is a contradiction. \[ \square \]

**Theorem 2.6.** There does not exist $W_4$-flat super quasi Einstein manifold.

**Proof.** Suppose we have a $W_4$-flat super quasi Einstein manifold. Using equations (6) and (9), we get
\[ R(X, Y, \rho, \mu) = \frac{1}{n-1} \{ g(X, Y)S(\rho, \mu) - g(X, \rho)S(Y, \mu) \} \]
\[ = \frac{1}{n-1} \{ \gamma g(X, Y) - A(X) [\alpha B(Y) \]
\[ + \gamma A(Y) + \delta D(Y, \mu)] \} \]
\[ = \frac{1}{n-1} \{ -A(X) [\alpha B(Y) + \gamma A(Y) + \delta D(\mu, Y)] \} \quad (24) \]
\[ R(\rho, \mu, X, Y) = \frac{1}{n-1} \{ g(\rho, \mu)S(X, Y) - g(\rho, X)S(\mu, Y) \} \]
\[ = \frac{1}{n-1} \{ -A(X) [\alpha B(Y) + \gamma A(Y) + \delta D(\mu, Y)] \} . \quad (25) \]

Since $D$ is symmetric and $R(X, Y, \rho, \mu) = R(\rho, \mu, X, Y)$, equations (24) and (25) imply
\[ \gamma g(X, Y) = 0, \text{ for all } X, Y \in TM. \]
This implies $g(X, Y) = 0, \forall X, Y$, since $\gamma \neq 0$. This is a contradiction. \[ \square \]
Theorem 2.7. There does not exist $W_4$-flat mixed super quasi Einstein manifold.

Proof. Suppose we have a $W_4$-flat mixed super quasi Einstein manifold. Using equations (7) and (9), we yield

$$R(X, Y, \rho, \mu) = \frac{1}{n-1} \{ g(X, Y)S(\rho, \mu) - g(X, \rho)S(Y, \mu) \}$$

$$= \frac{1}{n-1} \{ \delta g(X, Y) - A(X) [ (\alpha + \gamma)B(Y) + \delta A(Y) + \epsilon D(Y, \mu) ] \}$$

(26)

$$R(\rho, \mu, X, Y) = \frac{1}{n-1} \{ g(\rho, \mu)S(X, Y) - g(\rho, X)S(\mu, Y) \}$$

$$= \frac{1}{n-1} \{ -A(X) [ (\alpha + \gamma)B(Y) + \delta A(Y) + \epsilon D(\mu, Y) ] \}$$

(27)

Since $D$ is symmetric and $R(X, Y, \rho, \mu) = R(\rho, \mu, X, Y)$, equations (26) and (27) imply

$$\delta g(X, Y) = 0,$$

for all $X, Y \in TM$.

But $\delta \neq 0$, so $g(X, Y) = 0$, for all $X, Y \in TM$. This is a contradiction. \(\square\)

Acknowledgement

D.T. Leyson was funded by Erasmus Mundus Mobility with Asia (EMMAsia) in the framework of the EU Erasmus Mundus Action 2. R.S. Lemece and D.T. Leyson were funded by the University of the Philippines Natural Sciences Research Institute (UP NSRI).

References


Received: February 11, 2014