A Note on Identities of Symmetry for
Generalized Carlitz’s $q$-Bernoulli Polynomials\(^1\)

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Abstract

In this paper, we investigate some symmetric properties of $p$-adic $q$-integral on $\mathbb{Z}_p$. A question was asked in [10] as to finding formulae of symmetries for the generalized Carlitz $q$-Bernoulli polynomials. From our investigation, we derive some

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new identities of symmetry for the generalized Carlitz $q$-Bernoulli polynomials which are a partial answer to that question.

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1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = p^{-1}$. Let $q$ be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < p^{-\frac{1}{p-1}}$. Let $d$ be a fixed positive integer. We set

$$X = \lim_{N \to \infty} \left( \mathbb{Z}/dp^N \mathbb{Z} \right), \quad X^* = \bigcup_{0 < a < dp \atop (a,p) = 1} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{ x \in X | x \equiv a \pmod{dp^N} \}, \quad (N \in \mathbb{N}),$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, (see [1-19]).

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (1.1)$$

where $[x]_q = \frac{1-q^x}{1-q}$.

From (1.1), we note that

$$qI_q(f_1) = I_q(f) + (q-1)f(0) + \frac{q-1}{\log q} f'(0), \quad (1.2)$$

where $f_1(x) = f(x + 1)$.

By (1.2), we easily get

$$q^n I_q(f_n) = I_q(f) + (q-1) \sum_{l=0}^{n-1} f(l) + \frac{q-1}{\log q} \sum_{l=0}^{n-1} f'(l), \quad (1.3)$$

where $n \in \mathbb{N}$ and $f_n(x) = f(x + n)$.

It is not difficult to show that

$$\int_X f(x) d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x), \quad \text{(see [9])},$$

where $f \in UD(\mathbb{Z}_p)$. 
The Bernoulli polynomials are defined by the generating function to be
\[
\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see} \ [14, 15, 16, 17, 18]). \tag{1.4}
\]
When \( x = 0 \), \( B_n = B_n(0) \) is called the \( n \)-th Bernoulli number.
By (1.4), we easily get
\[
B_0 = 1 \quad \text{and} \quad (B + 1)^n - B_n = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}
\]
with the usual convention about replacing \( B_i \) by \( B_i \) (see [18, 19]).
In [3], Carlitz considered the \( q \)-extensions of Bernoulli numbers as follows:
\[
\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases}
\tag{1.5}
\]
He also defined \( q \)-Bernoulli polynomials as follows:
\[
\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^lx \beta_{l,q}, \quad (\text{see} \ [2, 3]). \tag{1.6}
\]
Recently, Kim gave the Witt’s formula for the Carlitz’s \( q \)-Bernoulli polynomials which are given by
\[
\int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0), \quad (\text{see} \ [9]). \tag{1.7}
\]
When \( x = 0 \), \( \beta_{n,q} = \beta_{n,q}(0) \) is called the \( n \)-th Carlitz \( q \)-Bernoulli number.
From (1.2) and (1.7), we note that
\[
q\beta_{n,q}(1) - \beta_{n,q} = \begin{cases} q - 1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases} \tag{1.8}
\]
By (1.7), we get
\[
\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(x) = \sum_{l=0}^{n} \binom{n}{l} q^lx \int_{\mathbb{Z}_p} [y]_q^l d\mu_q(x) [x]_q^{n-l} \tag{1.9}
\]
\[
= \sum_{l=0}^{n} \binom{n}{l} q^lx \beta_{l,q} [x]_q^{n-l} = \left( q^x \beta_q + [x]_q \right)^n.
\]
Let \( \chi \) be a primitive Dirichlet character with conductor \( d \in \mathbb{Z}_{\geq 0} \), with \( (d, p) = 1 \). Then the generalized Bernoulli polynomials attached to \( \chi \) are defined by the generating function to be
\[
\frac{t}{e^{at} - 1} \left( \sum_{\chi(a)} \chi(a) e^{at} \right) e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}. \tag{1.10}
\]
When \( x = 0 \), \( B_{n,\chi} = B_{n,\chi}(0) \) is called the \( n \)-th generalized Bernoulli number attached to \( \chi \) (see [9, 17, 18]). By (1.10), we get

\[
B_{n,\chi}(x) = d^{k-1} \sum_{a=0}^{d-1} \chi(a) B_k \left( \frac{a + x}{d} \right), \quad (k \geq 0).
\]  

(1.11)

In [9], the \( q \)-extension of (1.11) is given by

\[
\beta_{n,\chi,q}(x) = [d]_q^{n-1} \sum_{a=0}^{d-1} \chi(a) q^a \beta_{n,q^d} \left( \frac{a + x}{d} \right),
\]  

(1.12)

where \( \beta_{n,\chi,q}(x) \) are called the generalized \( q \)-Bernoulli polynomials attached to \( \chi \).

From (1.1) and (1.12), we note that

\[
\beta_{n,\chi,q}(x) = \int_X [x + y]^n_q \chi(y) d\mu_q(y), \quad (n \geq 0), \quad \text{(see [10])}.
\]  

(1.13)

When \( x = 0 \), \( \beta_{n,\chi,q} = \beta_{n,\chi,q}(0) \) is called the \( n \)-th generalized Carlitz \( q \)-Bernoulli number attached to \( \chi \).

Indeed, by (1.13), we get

\[
\int_X [x + y]^n_q \chi(y) d\mu_q(y) = \frac{1}{[d]_q} \sum_{a=0}^{d-1} \chi(a) q^a \int_{\mathbb{Z}_p} [x + a + dy]^n_q d\mu_{q^d}(y)
\]

\[
= [d]_q^{n-1} \sum_{a=0}^{d-1} \chi(a) q^a \int_{\mathbb{Z}_p} \left[ \frac{x + a}{d} + y \right]^n_{q^d} d\mu_{q^d}(y)
\]

\[
= [d]_q^{n-1} \sum_{a=0}^{d-1} \chi(a) q^a \beta_{n,q^d} \left( \frac{x + a}{d} \right).
\]

In this paper, we investigate some symmetric properties of \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \). A question was asked in [10] as to finding formulae of symmetries for the generalized Carlitz \( q \)-Bernoulli polynomials. From our investigation, we derive some new identities of symmetry for the generalized Carlitz \( q \)-Bernoulli polynomials which are a partial answer to that question.

2. Symmetric identities of generalized \( q \)-Bernoulli polynomials

From (1.13), we note that

\[
\sum_{n=0}^{\infty} \beta_{n,\chi,q}(x) \frac{t^n}{n!} = \int_X \chi(y) e^{[x+y]_q t} d\mu_q(y).
\]  

(2.1)

Let \( w_1, w_2 \) be natural numbers.
Then, by (2.1), we get
\[
\frac{1}{[w_1]_q} \int_X \chi (y) e^{[w_1 w_2 x + w_2 j + w_1 y]_q} t d \mu_{w_1} (y) \tag{2.2}
\]
\[
= \lim_{N \to \infty} \frac{1}{[dw_1 w_2 p^N]_q} \sum_{i=0}^{dw_2-1} \sum_{j=0}^{dw_1-1} e^{[w_1 w_2 x + w_2 j + w_1 (i + dw_2 y)]_q} t q^{dw_1 w_2 y}.
\]
Thus, from (2.2), we have
\[
\frac{1}{[w_1]_q} \sum_{j=0}^{dw_1-1} \chi (j) q^{w_2 j} \int_X \chi (y) e^{[w_1 w_2 x + w_2 j + w_1 y]_q} t d \mu_{w_1} (y) \tag{2.3}
\]
\[
= \lim_{N \to \infty} \frac{1}{[dw_1 w_2 p^N]_q} \sum_{y=0}^{p^N-1} \sum_{j=0}^{dw_1-1} \sum_{i=0}^{dw_2-1} \chi (i) \chi (j) \times q^{w_1 i + w_2 j + dw_1 w_2 y} e^{[w_1 w_2 x + w_2 j + w_1 (i + dw_2 y)]_q} t.
\]
By the same method as (2.3), we get
\[
\frac{1}{[w_2]_q} \sum_{j=0}^{dw_2-1} \chi (j) q^{w_1 j} \int_X \chi (y) e^{[w_1 w_2 x + w_2 j + w_2 y]_q} t d \mu_{w_2} (y) \tag{2.4}
\]
\[
= \lim_{N \to \infty} \frac{1}{[dw_1 w_2 p^N]_q} \sum_{y=0}^{p^N-1} \sum_{j=0}^{dw_2-1} \sum_{i=0}^{dw_1-1} \chi (i) \chi (j) \times q^{w_2 i + w_1 j + dw_1 w_2 y} e^{[w_1 w_2 x + w_1 j + w_2 (i + dw_1 y)]_q} t.
\]
Therefore, by (2.3) and (2.4), we obtain the following theorem.

**Theorem 2.1.** For $w_1, w_2 \in \mathbb{N}$, we have
\[
\frac{1}{[w_1]_q} \sum_{j=0}^{dw_1-1} \chi (j) q^{w_2 j} \int_X \chi (y) e^{[w_1 w_2 x + w_2 j + w_1 y]_q} t d \mu_{w_1} (y)
\]
\[
= \frac{1}{[w_2]_q} \sum_{j=0}^{dw_2-1} \chi (j) q^{w_1 j} \int_X \chi (y) e^{[w_1 w_2 x + w_1 j + w_2 y]_q} t d \mu_{w_2} (y). \tag{2.5}
\]
Note that
\[
[w_1 w_2 x + w_2 j + w_1 y]_q = [w_1]_q \left[ w_2 x + \frac{w_2 j}{w_1} + y \right]_{q^{w_1}} \tag{2.5}
\]
and
\[
[w_1 w_2 x + w_1 j + w_2 y]_q = [w_2]_q \left[ w_1 x + \frac{w_1 j}{w_2} + y \right]_{q^{w_2}}. \tag{2.6}
\]
Therefore, by Theorem 2.1, (2.5) and (2.6), we obtain the following corollary.
Remark. We note that Theorem 2.3 is a partial answer to Question 1 in [10].

**Corollary 2.2.** For $n \geq 0$, we have

$$
[w_1]_{q}^{n-1} \sum_{j=0}^{d w_1-1} \chi (j) q^{w_2 j} \int_{X} \chi (y) \left[w_2 x + \frac{w_2}{w_1} j + y\right]^{n}_{q^{w_1}} d \mu_{q^{w_1}} (y)
$$

$$
= [w_2]_{q}^{n-1} \sum_{j=0}^{d w_2-1} \chi (j) q^{w_1 j} \int_{X} \chi (y) \left[w_1 x + \frac{w_1}{w_2} j + y\right]^{n}_{q^{w_2}} d \mu_{q^{w_2}} (y).
$$

Therefore, by (1.13) and Corollary 2.2, we obtain the following theorem.

**Theorem 2.3.** For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$, we have

$$
[w_1]_{q}^{n-1} \sum_{j=0}^{d w_1-1} \chi (j) q^{w_2 j} \beta_{n, \chi, q^{w_1}} \left(w_2 x + \frac{w_2}{w_1} j\right)
$$

$$
= [w_2]_{q}^{n-1} \sum_{j=0}^{d w_2-1} \chi (j) q^{w_1 j} \beta_{n, \chi, q^{w_2}} \left(w_1 x + \frac{w_1}{w_2} j\right).
$$

**Remark.** We note that Theorem 2.3 is a partial answer to Question 1 in [10].

From (1.13), we can derive the following equation (2.7):

$$
\int_{X} \chi (y) \left[w_2 x + \frac{w_2}{w_1} j + y\right]^{n}_{q^{w_1}} d \mu_{q^{w_1}} (y) (2.7)
$$

$$
= \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \sum_{j=0}^{d w_1-1} \chi (j) \left[w_2 x + \frac{w_2}{w_1} j + y\right]^{n-i}_{q^{w_1}} d \mu_{q^{w_1}} (y) \right) (2.8)
$$

Thus, by (2.7), we get

$$
[w_1]_{q}^{n-1} \sum_{j=0}^{d w_1-1} \chi (j) q^{w_2 j} \int_{X} \chi (y) \left[w_2 x + \frac{w_2}{w_1} j + y\right]^{n}_{q^{w_1}} d \mu_{q^{w_1}} (y)
$$

$$
= \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) \left[w_1]_{q}^{n-i-1} [w_2]_{q}^{i} \left( \sum_{j=0}^{d w_1-1} \chi (j) \left[w_2 x + \frac{w_2}{w_1} j + y\right]^{n-i}_{q^{w_1}} d \mu_{q^{w_1}} (w_2 x) \right) \right.
$$

$$
= \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_1]_{q}^{n-i-1} [w_2]_{q}^{n-i} \left( \sum_{j=0}^{d w_1-1} \chi (j) \left[w_2 x + \frac{w_2}{w_1} j + y\right]^{n-i}_{q^{w_1}} d \mu_{q^{w_1}} (w_2 x) \right)
$$

$$
= \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) [w_1]_{q}^{n-i-1} [w_2]_{q}^{n-i} T_{n,i} (dw_1, q^{w_2} | \beta_{i, \chi, q^{w_1}} (w_2 x),
$$
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where

$$T_{n,i}(w, q|\chi) = \sum_{j=0}^{w-1} [j]_q^{n-i} q^{j(i+1)} \chi(j). \quad (2.9)$$

By the same method as (2.8), we get

$$[w]_q^{n-1} \sum_{j=0}^{dw_2-1} \chi(j) q^{w_1j} \int_x^y \chi(y) \left[ w_1x + \frac{w_1}{w_2}j + y \right]_q^n d\mu_{w_2}(y) = \sum_{i=0}^{n} \binom{n}{i} [w_2]_{q}^{i-1} [w_1]_{q}^{n-i} T_{n,i}(dw_2, q^{w_1}|\chi) \beta_{i,n,w_2}(w_1x). \quad (2.10)$$

Therefore, by (2.8), (2.9) and (2.10), we obtain the following theorem.

**Theorem 2.4.** For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$, we have

$$\sum_{i=0}^{n} \binom{n}{i} [w_1]_{q}^{i-1} [w_2]_{q}^{n-i} T_{n,i}(dw_1, q^{w_2}|\chi) \beta_{i,n,w_1}(w_2x) = \sum_{i=0}^{n} \binom{n}{i} [w_2]_{q}^{i-1} [w_1]_{q}^{n-i} T_{n,i}(dw_2, q^{w_1}|\chi) \beta_{i,n,w_2}(w_1x),$$

where $T_{n,i}(w, q|\chi) = \sum_{j=0}^{w-1} [j]_q^{n-i} q^{j(i+1)} \chi(j)$.

**Remark.**

1. Let $\chi$ be the trivial character. Then we have $\beta_{n,\chi_{\text{triv}},q^{w_1}}(w_2x) = \beta_{n,\chi,q^{w_1}}(w_2x)$, $(n \geq 0)$.

2. For $\chi = \chi_{\text{triv}}$, we have

$$\sum_{i=0}^{n} \binom{n}{i} [w_1]_{q}^{i-1} [w_2]_{q}^{n-i} T_{n,i}(w_1, q^{w_2}) \beta_{i,n,w_2}(w_2x) = \sum_{i=0}^{n} \binom{n}{i} [w_2]_{q}^{i-1} [w_1]_{q}^{n-i} T_{n,i}(w_2, q^{w_1}) \beta_{i,n,w_2}(w_1x),$$

where $T_{n,i}(w, q) = \sum_{j=0}^{w-1} [j]_q^{n-i} q^{j(i+1)}$.

3. We note that Theorem 2.4 is another partial answer to Question 1 in [10].

**References**


