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A Note on Identities of Symmetry for Generalized Carlitz's q-Bernoulli Polynomials[1]

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Abstract

In this paper, we investigate some symmetric properties of p-adic q-integral on $\mathbb{Z}_p$. A question was asked in [10] as to finding formulae of symmetries for the generalized Carlitz q-Bernoulli polynomials. From our investigation, we derive some

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new identities of symmetry for the generalized Carlitz q-Bernoulli polynomials which are a partial answer to that question.

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1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = p^{-1}$. Let q be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, we assume that $|1-q|_p < p^{-\frac{1}{p-1}}$. Let d be a fixed positive integer. We set

$$X = \varprojlim_{N} \left(\mathbb{Z}/dp^N\mathbb{Z}\right), \qquad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \left\{ x \in X \,|\, x \equiv a \pmod{dp^N} \right\}, \quad (N \in \mathbb{N}),$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$, (see [1-19]).

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)\, q^x, \tag{1.1}$$

where $[x]_q = \frac{1-q^x}{1-q}$,

From (1.1), we note that

$$qI_q(f_1) = I_q(f) + (q-1) f(0) + \frac{q-1}{\log q} f'(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$.

By (1.2), we easily get

$$q^n I_q(f_n) = I_q(f) + (q-1)\sum_{l=0}^{n-1} f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} f'(l), \tag{1.3}$$

where $n \in \mathbb{N}$ and $f_n(x) = f(x+n)$.

It is not difficult to show that

$$\int_X f(x)\, d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x), \quad (\text{see [9]}),$$

where $f \in UD(\mathbb{Z}_p)$.

The Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (\text{see [14, 15, 16, 17, 18]}). \tag{1.4}$$

When $x=0$, $B_n=B_n(0)$ is called the n-th Bernoulli number.

By (1.4), we easily get

$$B_0=1 \text{ and } (B+1)^n-B_n=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n>1,\end{cases}$$

with the usual convention about replacing B^i by B_i (see [18, 19]).

In [3], Carlitz considered the q-extensions of Bernoulli numbers as follows :

$$\beta_{0,q}=1,\qquad q(q\beta_q+1)^n-\beta_{n,q}=\begin{cases}1, & \text{if } n=1\\ 0, & \text{if } n>1,\end{cases} \tag{1.5}$$

with the usual convention about replacing β_q^i by $\beta_{i,q}$.

He also defined q-Bernoulli polynomials as follows :

$$\beta_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\beta_{l,q},\quad (\text{see [2, 3]}). \tag{1.6}$$

Recently, Kim gave the Witt's formula for the Carlitz's q-Bernoulli polynomials which are given by

$$\int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_q(y)=\beta_{n,q}(x),\ (n\geq 0),\quad (\text{see [9]}). \tag{1.7}$$

When $x=0$, $\beta_{n,q}=\beta_{n,q}(0)$ is called the n-th Carlitz q-Bernoulli number.

From (1.2) and (1.7), we note that

$$q\beta_{n,q}(1)-\beta_{n,q}=\begin{cases}q-1 & \text{if } n=0\\ 1 & \text{if } n=1\\ 0 & \text{if } n>1.\end{cases} \tag{1.8}$$

By (1.7), we get

$$\begin{aligned}\beta_{n,q}(x)&=\int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_q(x)=\sum_{l=0}^{n}\binom{n}{l}q^{lx}\int_{\mathbb{Z}_p}[y]_q^l\,d\mu_q(x)[x]_q^{n-l}\\ &=\sum_{l=0}^{n}\binom{n}{l}q^{lx}\beta_{l,q}[x]_q^{n-l}=\left(q^x\beta_q+[x]_q\right)^n.\end{aligned} \tag{1.9}$$

Let χ be a primitive Dirichlet character with conductor $d\in\mathbb{Z}_{\geq 0}$, with $(d,p)=1$. Then the generalized Bernoulli polynomials attached to χ are defined by the generating funciton to be

$$\frac{t}{e^{dt}-1}\left(\sum_{a=0}^{d-1}\chi(a)e^{at}\right)e^{xt}=\sum_{n=0}^{\infty}B_{n,\chi}(x)\frac{t^n}{n!}. \tag{1.10}$$

When $x=0$, $B_{n,\chi}=B_{n,\chi}(0)$ is called the n-th generalized Bernoulli number attached to χ (see [9, 17, 18]). By (1.10), we get

$$B_{k,\chi}(x)=d^{k-1}\sum_{a=0}^{d-1}\chi(a)\,B_k\left(\frac{a+x}{d}\right),\quad (k\geq 0). \tag{1.11}$$

In [9], the q-extension of (1.11) is given by

$$\beta_{n,\chi,q}(x)=[d]_q^{n-1}\sum_{a=0}^{d-1}\chi(a)\,q^a\beta_{n,q^d}\left(\frac{a+x}{d}\right), \tag{1.12}$$

where $\beta_{n,\chi,q}(x)$ are called the generalized q-Bernoulli polynomials attached to χ.

From (1.1) and (1.12), we note that

$$\beta_{n,\chi,q}(x)=\int_X [x+y]_q^n\,\chi(y)\,d\mu_q(y),\quad (n\geq 0),\quad (\text{see [10]}). \tag{1.13}$$

When $x=0$, $\beta_{n,\chi,q}=\beta_{n,\chi,q}(0)$ is called the n-th generalized Carlitz q-Bernoulli number attached to χ.

Indeed, by (1.13), we get

$$\begin{aligned}\int_X [x+y]_q^n\,\chi(y)\,d\mu_q(y)&=\frac{1}{[d]_q}\sum_{a=0}^{d-1}\chi(a)\,q^a\int_{\mathbb{Z}_p}[x+a+dy]_q^n\,d\mu_{q^d}(y)\\&=[d]_q^{n-1}\sum_{a=0}^{d-1}\chi(a)\,q^a\int_{\mathbb{Z}_p}\left[\frac{x+a}{d}+y\right]_{q^d}^n d\mu_{q^d}(y)\\&=[d]_q^{n-1}\sum_{a=0}^{d-1}\chi(a)\,q^a\beta_{n,q^d}\left(\frac{x+a}{d}\right).\end{aligned}$$

In this paper, we investigate some symmetric properties of p-adic q-integral on $\mathbb{Z}_p$. A question was asked in [10] as to finding formulae of symmetries for the generalized Carlitz q-Bernoulli polynomials. From our investigation, we derive some new identities of symmetry for the generalized Carlitz q-Bernoulli polynomials which are a partial answer to that question.

2. Symmetric identities of generalized q-Bernoulli polynomials

From (1.13), we note that

$$\sum_{n=0}^{\infty}\beta_{n,\chi,q}(x)\frac{t^n}{n!}=\int_X \chi(y)\,e^{[x+y]_q t}d\mu_q(y). \tag{2.1}$$

Let w_1, w_2 be natural numbers.

Then, by (2.1), we get

$$\frac{1}{[w_1]_q}\int_X \chi(y)\, e^{[w_1w_2x+w_2j+w_1y]_q t} d\mu_{q^{w_1}}(y) \tag{2.2}$$
$$=\lim_{N\to\infty}\frac{1}{[dw_1w_2p^N]_q}\sum_{i=0}^{dw_2-1}\chi(i)\, q^{w_1 i}\sum_{y=0}^{p^N-1} e^{[w_1w_2x+w_2j+w_1(i+dw_2y)]_q t} q^{dw_1w_2y}.$$

Thus, from (2.2), we have

$$\frac{1}{[w_1]_q}\sum_{j=0}^{dw_1-1}\chi(j)\, q^{w_2 j}\int_X \chi(y)\, e^{[w_1w_2x+w_2j+w_1y]_q t} d\mu_{q^{w_1}}(y) \tag{2.3}$$
$$=\lim_{N\to\infty}\frac{1}{[dw_1w_2p^N]_q}\sum_{y=0}^{p^N-1}\sum_{j=0}^{dw_1-1}\sum_{i=0}^{dw_2-1}\chi(i)\,\chi(j)$$
$$\times\, q^{w_1 i+w_2 j+dw_1w_2y} e^{[w_1w_2x+w_2j+w_1(i+dw_2y)]_q t}.$$

By the same method as (2.3), we get

$$\frac{1}{[w_2]_q}\sum_{j=0}^{dw_2-1}\chi(j)\, q^{w_1 j}\int_X \chi(y)\, e^{[w_1w_2x+w_1j+w_2y]_q t} d\mu_{q^{w_2}}(y) \tag{2.4}$$
$$=\lim_{N\to\infty}\frac{1}{[dw_1w_2p^N]_q}\sum_{y=0}^{p^N-1}\sum_{j=0}^{dw_2-1}\sum_{i=0}^{dw_1-1}\chi(i)\,\chi(j)$$
$$\times\, q^{w_2 i+w_1 j+dw_1w_2y} e^{[w_1w_2x+w_1j+w_2(i+dw_1y)]_q t}.$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *For w_1, $w_2 \in \mathbb{N}$, we have*

$$\frac{1}{[w_1]_q}\sum_{j=0}^{dw_1-1}\chi(j)\, q^{w_2 j}\int_X \chi(y)\, e^{[w_1w_2x+w_2j+w_1y]_q t} d\mu_{q^{w_1}}(y)$$
$$=\frac{1}{[w_2]_q}\sum_{j=0}^{dw_2-1}\chi(j)\, q^{w_1 j}\int_X \chi(y)\, e^{[w_1w_2x+w_1j+w_2y]_q t} d\mu_{q^{w_2}}(y).$$

Note that

$$[w_1w_2x+w_2j+w_1y]_q=[w_1]_q\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}} \tag{2.5}$$

and

$$[w_1w_2x+w_1j+w_2y]_q=[w_2]_q\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}. \tag{2.6}$$

Therefore, by Theorem 2.1, (2.5) and (2.6), we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$[w_1]_q^{n-1} \sum_{j=0}^{dw_1-1} \chi(j) q^{w_2 j} \int_X \chi(y) \left[w_2 x + \frac{w_2}{w_1} j + y\right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y)$$
$$= [w_2]_q^{n-1} \sum_{j=0}^{dw_2-1} \chi(j) q^{w_1 j} \int_X \chi(y) \left[w_1 x + \frac{w_1}{w_2} j + y\right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y).$$

Therefore, by (1.13) and Corollary 2.2, we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, w_1, $w_2 \in \mathbb{N}$, we have*

$$[w_1]_q^{n-1} \sum_{j=0}^{dw_1-1} \chi(j) q^{w_2 j} \beta_{n,\chi,q^{w_1}} \left(w_2 x + \frac{w_2}{w_1} j\right)$$
$$= [w_2]_q^{n-1} \sum_{j=0}^{dw_2-1} \chi(j) q^{w_1 j} \beta_{n,\chi,q^{w_2}} \left(w_1 x + \frac{w_1}{w_2} j\right).$$

Remark. We note that Theorem 2.3 is a partial answer to Question 1 in [10].

From (1.13), we can derive the following equation (2.7) :

$$\int_X \chi(y) \left[w_2 x + \frac{w_2}{w_1} j + y\right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y) \tag{2.7}$$
$$= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]_{q^{w_2}}^i q^{w_2(n-i)j} \int_X [w_2 x + y]_{q^{w_1}}^{n-i} \chi(y) d\mu_{q^{w_1}}(y)$$
$$= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]_{q^{w_2}}^i q^{w_2(n-i)j} \beta_{n-i,\chi,q^{w_1}}(w_2 x).$$

Thus, by (2.7), we get

$$[w_1]_q^{n-1} \sum_{j=0}^{dw_1-1} \chi(j) q^{w_2 j} \int_X \chi(y) \left[w_2 x + \frac{w_2}{w_1} j + y\right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y) \tag{2.8}$$
$$= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{n-i-1} [w_2]_q^i \left(\sum_{j=0}^{dw_1-1} [j]_{q^{w_2}}^i q^{w_2 j(n-i+1)} \chi(j)\right) \beta_{n-i,\chi,q^{w_1}}(w_2 x)$$
$$= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{i-1} [w_2]_q^{n-i} \left(\sum_{j=0}^{dw_1-1} [j]_{q^{w_2}}^{n-i} q^{w_2 j(i+1)} \chi(j)\right) \beta_{i,\chi,q^{w_1}}(w_2 x)$$
$$= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{i-1} [w_2]_q^{n-i} T_{n,i}(dw_1, q^{w_2}|\chi) \beta_{i,\chi,q^{w_1}}(w_2 x),$$

where

$$T_{n,i}(w,\, q|\chi) = \sum_{j=0}^{w-1} [j]_q^{n-i}\, q^{j(i+1)} \chi(j)\,. \tag{2.9}$$

By the same method as (2.8), we get

$$[w_2]_q^{n-1} \sum_{j=0}^{dw_2-1} \chi(j)\, q^{w_1 j} \int_X \chi(y) \left[w_1 x + \frac{w_1}{w_2} j + y \right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y) \tag{2.10}$$

$$= \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^{i-1} [w_1]_q^{n-i}\, T_{n,i}(dw_2,\, q^{w_1}|\chi)\, \beta_{i,\chi,q^{w_2}}(w_1 x)\,.$$

Therefore, by (2.8), (2.9) and (2.10), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, $w_1,\, w_2 \in \mathbb{N}$, we have*

$$\sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{i-1} [w_2]_q^{n-i}\, T_{n,i}(dw_1,\, q^{w_2}|\chi)\, \beta_{i,\chi,q^{w_1}}(w_2 x)$$

$$= \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^{i-1} [w_1]_q^{n-i}\, T_{n,i}(dw_2,\, q^{w_1}|\chi)\, \beta_{i,\chi,q^{w_2}}(w_1 x)\,,$$

where $T_{n,i}(w,\, q|\chi) = \sum_{j=0}^{w-1} [j]_q^{n-i}\, q^{j(i+1)} \chi(j)\,.$

Remark. (1) Let χ be the trivial character. Then we have $\beta_{n,\chi_{\text{triv}},q^{w_1}}(w_2 x) = \beta_{n,q^{w_1}}(w_2 x)$, $(n \geq 0)$.

(2) For $\chi = \chi_{\text{triv}}$, we have

$$\sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{i-1} [w_2]_q^{n-i}\, T_{n,i}(w_1,\, q^{w_2})\, \beta_{i,q^{w_1}}(w_2 x)$$

$$= \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^{i-1} [w_1]_q^{n-i}\, T_{n,i}(w_2,\, q^{w_1})\, \beta_{i,q^{w_2}}(w_1 x)\,,$$

where $T_{n,i}(w, q) = \sum_{j=0}^{w-1} [j]_q^{n-i}\, q^{j(i+1)}$.

(3) We note that Theorem 2.4 is another partial answer to Question 1 in [10].

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