New Iterative Algorithm for Variational Inequality Problem and Fixed Point Problem in Hilbert Spaces

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Abstract

In this paper, we introduce a new iterative scheme with a countable family of nonexpansive mappings for variational inequality problem in a Hilbert space and prove a strong convergence theorem for the iterative scheme.

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1. INTRODUCTION

Let H be a Hilbert space and C be a nonempty closed convex subset of H . Let $F : H \rightarrow H$ be a nonlinear operator. The variational inequality problem on F is to find a point $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

The variational inequality problem is denoted by $VI(F, C)$ [5].

F is called to be κ -Lipschitzian and η -strongly monotone if there exist constants $\kappa, \eta > 0$ such that

$$\|Fx - Fy\| \leq \kappa\|x - y\|, \quad \text{for all } x, y \in H,$$

and

$$\langle Fx - Fy, x - y \rangle \geq \eta\|x - y\|^2, \quad \text{for all } x, y \in H,$$

respectively.

It is well known that if F is strongly monotone and Lipschitzian on C , then $VI(F, C)$ has a unique solution. An important problem is how to find a solution of $VI(F, C)$.

It is known that the $VI(F, C)$ is equivalent to the fixed point equation [16]

$$u^* = P_C(u^* - \mu F(u^*)), \quad (1.1)$$

where P_C is the projection from H onto C ; i.e.,

$$P_C x = \min_{y \in C} \|x - y\|, \quad \forall x \in H,$$

and where $\mu > 0$ is an arbitrarily fixed constant. So, the fixed point methods can be implemented to find a solution of the $VI(F, C)$.

The fixed point formulation (1.1) involves the projection P_C , which may not be easy to compute, due to the complexity of the convex set C . Hence, in order to reduce the complexity probably caused by the projection P_C , Yamada [16] introduced the hybrid steepest-descent method for solving the $VI(F, C)$ by replacing P_C with a nonexpansive mapping T . Recall that a mapping $T : H \rightarrow H$ is called nonexpansive if for all $x, y \in H$, one holds

$$\|Tx - Ty\| \leq \|x - y\|.$$

The set of fixed points of T is denoted by $Fix(T)$. More precisely, Yamada [16] gave the following iterative scheme:

$$u_0 \in H, \quad u_{n+1} = Tu_n - \lambda_{n+1}\mu F(Tu_n), \quad n \geq 0, \quad (1.2)$$

where $T : H \rightarrow H$ is a nonexpansive mapping, F is η -strongly monotone and κ -Lipschitzian on $C = F(T)$, $\{\lambda_n\}$ is a sequence in $(0, 1)$ and μ is a fixed number with $0 < \mu < 2\eta/\kappa^2$. Yamada [16] proved that if the sequence $\{\lambda_n\}$ satisfies the conditions:

- (L1) $\lim_{n \rightarrow \infty} \lambda_n = 0$,
- (L2) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
- (L3) $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$,

then $\{u_n\}$ converges strongly to the unique solution of $VI(F, C)$.

Since Yamada's hybrid steepest-descent method for solving variational inequalities [16], there are much research on this aspect; see, e.g., [15, 19, 20, 21].

In 2011, Buong and Duong [2] introduced a new iterative algorithm, based on a combination of the hybrid steepest-descent method for variational inequalities with the Krasnosel'skii-Mann type algorithm for fixed point problems. Very recently Zhou and Wang [23] simplified the algorithm of Buong and Duong and proved a strong convergence theorem for the variational inequality problem and fixed point problem on finite nonexpansive mappings in Hilbert space. In this paper, motivated by the work of Zhou and Wang [23], we introduce a new iterative algorithm for solving the solution of variational inequality problem and prove the solution is the common fixed point of a countable family of nonexpansive mappings in Hilbert space.

1. PRELIMINARIES

Let H be a Hilbert space. Let $T : H \rightarrow H$ be nonexpansive mapping and let $F : H \rightarrow H$ be a κ -Lipschitzian and η -strong monotone nonlinear operator.

Lemma 2.1 ([16]). *Assume $\lambda \in (0, 1)$ and $\mu \in (0, 2\eta/\kappa^2)$. Define a mapping $T^\lambda : H \rightarrow H$ by*

$$T^\lambda x = Tx - \lambda\mu F(Tx) \quad \forall x \in H.$$

Then $\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|$ for all $x, y \in H$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1)$.

Obviously, if $T = I$, then $T^\lambda = I - \lambda\mu F$ and by Lemma 2.1 we have

$$\|T^\lambda x - T^\lambda y\| = \|(I - \lambda\mu F)x - (I - \lambda\mu F)y\| \leq (1 - \lambda\tau)\|x - y\|$$

for all $x, y \in H$.

Lemma 2.2 ([7]). *Let $\{s_n\}, \{c_n\}$ be the sequences of nonnegative real numbers and let $\{a_n\} \subset (0, 1)$. Suppose $\{b_n\}$ is a real number sequence such that*

$$s_{n+1} \leq (1 - a_n)s_n + b_n + c_n, \quad n \geq 0.$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

- (1) *If $b_n \leq \beta a_n$ where $(\beta \geq 0)$, then $\{s_n\}$ is a bounded sequence.*
- (2) *If we have*

$$\sum_{n=0}^{\infty} a_n = \infty \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{b_n}{a_n} \leq 0,$$

then $\lim_{n \rightarrow \infty} s_n = 0$.

Let $T : H \rightarrow H$ be a nonexpansive mapping. Let $\alpha \in (0, 1)$. Define the mapping $S = \alpha I + (1 - \alpha)T$. Then the mapping S is said to be an averaged mapping with $\text{Fix}(T) = \text{Fix}(S)$. Furthermore, on a family of averaged mappings, we have the following conclusion.

Lemma 2.3 ([1, 12]). *Let $\{T_i\}_{i=1}^N$ be the averaged mappings on H and assume that $\cap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. Then $\cap_{i=1}^N \text{Fix}(T_i) = \text{Fix}(T_1 \dots T_N)$.*

Lemma 2.4 ([13]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following condition*

$$a_{n+1} \leq (1 - b_n)a_n + b_nc_n,$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers such that

$$b_n \in [0, 1], \sum_{n=0}^{\infty} b_n = \infty \text{ and } \limsup_{n \rightarrow \infty} c_n \leq 0.$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

3. MAIN RESULTS

Theorem 3.1. *Let H be a real Hilbert space and $F : H \rightarrow H$ be an L -Lipschitz continuous and η -strongly monotone mapping. Let $\{T_i\}_{i=1}^{\infty}$ be a countable family of nonexpansive of H such that $C = \cap_{i=1}^{\infty} \text{Fix}(T_i) \neq \emptyset$. Let $\{\lambda_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ be sequences in $(0, 1)$. Assume that $\{\beta_n\}$ is strictly decreasing and let $\beta_0 = 1$ and $\mu \in (0, 2\eta/L^2)$. Suppose that the following conditions hold:*

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \sum_{n=1}^{\infty} \lambda_n = \infty, \lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\lambda_n} = 1 \text{ and } \sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty.$$

Then the sequence $\{x_n\}$ generated by the following manner:

$$x_1 \in H, x_{n+1} = (I - \lambda_n \mu F)(\beta_n x_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \Pi_{j=1}^i U_j x_n), \quad \forall n \geq 1, \quad (3.1)$$

where $U_i = \alpha_i I + (1 - \alpha_i)T_i$, converges strongly to the unique solution x^* of the variational inequality:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (3.2)$$

Proof.

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(I - \lambda_n \mu F)(\beta_n x_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \Pi_{j=1}^i U_i x_n) - p\| \\
&= \|(I - \lambda_n \mu F)(\beta_n x_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \Pi_{j=1}^i U_i x_n) - (I - \lambda \mu F)p + \lambda_n \mu F(p)\| \\
&\leq \|(I - \lambda_n \mu F)(\beta_n x_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \Pi_{j=1}^i U_i x_n) - (I - \lambda \mu F)p\| + \lambda_n \mu \|F(p)\| \\
&\leq (1 - \tau \lambda_n) \|\beta_n x_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) \Pi_{j=1}^i U_i x_n - p\| + \lambda_n \mu \|F(p)\| \\
&\leq (1 - \tau \lambda_n) [\beta_n \|x_n - p\| + (1 - \beta_n) \|x_n - p\|] + \lambda_n \mu \|F(p)\| \\
&= (1 - \tau \lambda_n) \|x_n - p\| + \tau \lambda_n \frac{\mu}{\tau} \|F(p)\| \\
&\leq \max\{\|x_0 - p\|, \frac{\mu}{\tau} \|F(p)\|\}.
\end{aligned}$$

So, $\{x_n\}$ is bounded and so are $\{U_i x_n\}$ for each $i \geq 1$.

Next we prove that $x_{n+1} - x_n \rightarrow 0$ as $n \rightarrow \infty$. Let $y_n = \beta_n x_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) W_i x_n$, where $W_i = \Pi_{j=1}^i U_j$ for each $i \geq 1$. We rewrite (3.1) as follows:

$$x_{n+1} = (I - \lambda_n \mu F)y_n, \quad \forall n \geq 1, \quad (3.3)$$

Then

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|\beta_n x_n + \sum_{i=1}^n (\beta_{i-1} - \beta_i) W_i x_n - (\beta_{n-1} x_{n-1} + \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) W_i x_{n-1})\| \\
&= \|\beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1} + \sum_{i=1}^n (\beta_{i-1} - \beta_i) (W_i x_n - W_i x_{n-1}) \\
&\quad + (\beta_{n-1} - \beta_n) W_n x_{n-1}\| \\
&\leq \beta_n \|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| \|x_{n-1}\| + \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) \|W_i x_n - W_i x_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n| \|W_n x_{n-1}\| \\
&\leq \beta_n \|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| \|x_{n-1}\| + \sum_{i=1}^{n-1} (\beta_{i-1} - \beta_i) \|x_n - x_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n| \|W_n x_{n-1}\| \\
&= \beta_n \|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| (\|x_{n-1}\| + \|W_n x_{n-1}\|) + (1 - \beta_n) \|x_n - x_{n-1}\| \\
&= \|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| M,
\end{aligned}$$

where $M = \sup_{n \geq 1} \{\|x_{n-1}\| + \|W_n x_{n-1}\|\}$. Then

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(I - \lambda_n \mu F)y_n - (I - \lambda_{n-1} \mu F)y_{n-1}\| \\
&= \|(I - \lambda_n \mu F)(y_n - y_{n-1}) + ((I - \lambda_n \mu F) - (I - \lambda_{n-1} \mu F))y_{n-1}\| \\
&= \|(I - \lambda_n \mu F)(y_n - y_{n-1}) + (\lambda_{n-1} \mu F - \lambda_n \mu F)y_{n-1}\| \\
&= \|(I - \lambda_n \mu F)(y_n - y_{n-1}) + (\lambda_{n-1} - \lambda_n) \mu F y_{n-1}\| \\
&\leq (1 - \lambda_n \tau) \|y_n - y_{n-1}\| + |\lambda_{n-1} - \lambda_n| \mu \|F y_{n-1}\| \\
&\leq (1 - \lambda_n \tau) (\|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| M) + |\lambda_{n-1} - \lambda_n| \mu \|F y_{n-1}\| \\
&\leq (1 - \lambda_n \tau) \|x_n - x_{n-1}\| + |\beta_{n-1} - \beta_n| M + |\lambda_{n-1} - \lambda_n| \mu M',
\end{aligned}$$

where $M' = \sup_{n \geq 1} \{\|F y_n\|\}$. By Lemma 2.2 we conclude that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.4)$$

Since $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, it follows from (3.3) that

$$\|x_{n+1} - y_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.5)$$

Furthermore, by (3.4) and (3.5) we get

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.6)$$

Note that $y_n - x_n = \sum_{i=1}^n (\beta_{i-1} - \beta_i)(W_i x_n - x_n)$ and $\{\beta_n\}$ is a strictly decreasing sequence. So, for each $i \geq 1$, we have

$$\|W_i x_n - x_n\| \leq \sum_{i=1}^n (\beta_{i-1} - \beta_i) \|W_i x_n - x_n\| = \|y_n - x_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.7)$$

Next we prove that $\limsup_{n \rightarrow \infty} \langle -F(x^*), x_{n+1} - x^* \rangle \leq 0$. To prove this, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle -F(x^*), x_n - x^* \rangle = \lim_{i \rightarrow \infty} \langle -F(x^*), x_{n_i} - x^* \rangle.$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup \hat{x}$ for some $\hat{x} \in C$. By (3.7) and demiclosed principle we get $\hat{x} \in \text{Fix}(W_i)$ for each $i \geq 1$. Furthermore, by the definition of W_i and Lemma 2.3 we conclude that $\hat{x} \in \cap_{j=1}^{\infty} \text{Fix}(T_j)$. Since x^* is the unique solution of (3.2), we obtain

$$\limsup_{n \rightarrow \infty} \langle F(x^*), x^* - x_n \rangle = \lim_{i \rightarrow \infty} \langle F(x^*), x^* - \hat{x} \rangle \leq 0. \quad (3.8)$$

Finally, we prove that $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

By virtue of (3.3) and Lemma 2.1, we have

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
&= \|(I - \lambda_n \mu F)y_n - x^*\|^2 \\
&= \|(I - \lambda_n \mu F)y_n - (I - \lambda_n \mu F)x^* - \lambda_n \mu F(x^*)\|^2 \\
&= \|(I - \lambda_n \mu F)y_n - (I - \lambda_n \mu F)x^*\|^2 + 2\lambda_n \mu \langle F(x^*), (I - \lambda_n \mu F)x^* - (I - \lambda_n \mu F)y_n \rangle \\
&\quad + \lambda_n^2 \mu^2 \|F(x^*)\|^2 \\
&\leq (1 - \tau \lambda_n) \|x_n - x^*\|^2 + 2\lambda_n \mu \langle F(x^*), x^* - x_n \rangle + 2\lambda_n \mu \|F(x^*)\| \|x_n - y_n\| \\
&\quad + 2\lambda_n^2 \mu^2 (\|F(x^*)\| + \|F(y_n)\|) + \lambda_n^2 \mu^2 \|F(x^*)\|^2 \\
&= (1 - \tau \lambda_n) \|x_n - x^*\|^2 + 2\lambda_n \mu \langle F(x^*), x^* - x_n \rangle + 2\lambda_n \mu \|F(x^*)\| \|x_n - y_n\| \\
&\quad + \lambda_n^2 \mu^2 (2\|F(x^*)\| + 2\|F(y_n)\| + \|F(x^*)\|^2) \\
&\leq (1 - \tau \lambda_n) \|x_n - x^*\|^2 + \lambda_n \tau c_n,
\end{aligned}$$

where

$$c_n = \frac{\mu}{\tau} (2\langle F(x^*), x^* - x_n \rangle + 2\|F(x^*)\| \|x_n - y_n\| + \lambda_n \mu L)$$

and

$$L = \sup_{n \geq 1} (2\|F(x^*)\| + 2\|F(y_n)\| + \|F(x^*)\|^2) \text{ is a fixed constant.}$$

From (3.6)-(3.8), we know that $\limsup_{n \rightarrow \infty} c_n \leq 0$. By Lemma 2.4, we conclude that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. This completes the proof. $\square$

Corollary 3.1. *Let H be a real Hilbert space and $F : H \rightarrow H$ be an L -Lipschitz continuous and η -strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be a family of non-expansive of H such that $C = \cap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. Let $\{\lambda_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ be sequences in $(0, 1)$. Assume that $\mu \in (0, 2\eta/L^2)$. Suppose that the following conditions hold:*

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\lambda_n} = 1 \text{ and } \sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty.$$

Then the sequence $\{x_n\}$ generated by the following manner:

$$\left\{ \begin{array}{l} x_1 \in H, \\ x_2 = (I - \lambda_1 \mu F)(\beta_1 x_1 + (1 - \beta_1)U_1 x_1), \\ x_3 = (I - \lambda_2 \mu F)(\beta_2 x_2 + (1 - \beta_1)U_1 x_2 + (\beta_1 - \beta_2)U_1 U_2 x_2), \\ \vdots \\ x_{N+1} = (I - \lambda_N \mu F)(\beta_N x_N + (1 - \beta_1)U_1 x_N + (\beta_1 - \beta_2)U_1 U_2 x_N \\ \quad + \dots + (\beta_{N-1} - \beta_N)U_1 U_2 \dots U_N x_N), \\ x_{n+1} = (I - \lambda_n \mu F)(\beta_n x_n + (1 - \beta_1)U_1 x_n + (\beta_1 - \beta_2)U_1 U_2 x_n \\ \quad + \dots + (\beta_{N-2} - \beta_{N-1})U_1 U_2 \dots U_{N-1} x_n + (\beta_{N-1} - \beta_n)U_1 U_2 \dots U_N x_n), \quad \forall n > N, \end{array} \right.$$

where $U_i = \alpha_i I + (1 - \alpha_i)T_i$ ($i = 1, 2, \dots, s, N$), converges strongly to the unique solution x^* of the variational inequality:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

Corollary 3.2. Let H be a real Hilbert space and $F : H \rightarrow H$ be an L -Lipschitz continuous and η -strongly monotone mapping. Let T be a nonexpansive of H such that $C = \text{Fix}(T) \neq \emptyset$. Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be sequences in $(0, 1)$. Assume that $\mu \in (0, 2\eta/L^2)$. Suppose that the following conditions hold:

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\lambda_n} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| < \infty.$$

Then the sequence $\{x_n\}$ generated by the following manner:

$$x_1 \in H, \quad x_{n+1} = (I - \lambda_n \mu F)(\beta_n x_n + (1 - \beta_n)Ux_n), \quad \forall n \geq 1,$$

where $U = \alpha I + (1 - \alpha)T$ and $0 < \alpha < 1$, converges strongly to the unique solution x^* of the variational inequality:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

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